INTEGRAL GEOMETRY ON SURFACES OF CONSTANT NEGATIVE CURVATURE

BY L. A. SANTALÓ

1. **Introduction.** We use the expression "integral geometry" in the sense given it by Blaschke [4]. In a previous paper [11] we generalized to the sphere many formulas of plane integral geometry and at the same time applied these to the demonstration of certain inequalities referring to spherical curves.

The present paper considers analogous questions for surfaces of constant negative curvature and consequently for hyperbolic geometry [1].

In §§2–7 we define the measure of sets of geodesic lines and the cinematic measure, making application of both in order to obtain various integral formulas such as, for example, (4.6) which generalizes a classic result of Crofton for plane geometry and (7.5) which is the generalization of Blaschke's fundamental formula of plane integral geometry.

In §8 we apply the above results to the proof of the isoperimetric property of geodesic circles (inequality (8.4)). In §9 we obtain a sufficient condition that a convex figure be contained in the interior of another, thus generalizing to surfaces of constant negative curvature a result which H. Hadwiger [8] obtained for the plane.

For what follows we must remember that on the surfaces of curvature $K = -1$ the formulas of hyperbolic trigonometry are applicable [2; 638], that is, for a geodesic triangle of sides a, b, c and angles α, β, γ, we have

$$\begin{gathered}\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha,\\ \sinh a/\sin \alpha = \sinh b/\sin \beta = \sinh c/\sin \gamma, \qquad (1.1)\\ \sinh a \cos \beta = \cosh b \sinh c - \sinh b \cosh c \cos \alpha.\end{gathered}$$

2. **Measure of sets of geodesics.** Let us consider a surface of constant curvature $K = -1$. Let O be a fixed point on that surface and G a geodesic which does not pass through O. We know that through O passes only one geodesic perpendicular to G [6; 410]. Let v be the distance from O to G measured upon this perpendicular. We shall call θ the angle which the perpendicular geodesic makes with a fixed direction at O. We define as the "density" to measure sets of geodesics the differential expression

$$dG = \cosh v \, dv \, d\theta, \tag{2.1}$$

that is, the measure of a set of geodesics will be the integral of the expression (2.1) extended to this set.

Received March 12, 1943.