SEQUENCES OF STIELTJES INTEGRALS, III

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1. **Introduction.** The principal result of this paper is contained in Theorem 3.1 (§3). It can be considered as completing in the case of ordinary Riemann-Stieltjes integration the solution of the problem posed in [11; §1]. An immediate though non-trivial extension of this problem is also discussed here briefly (§5 and parts of §§7 and 8).

Another form of condition (3.5), the only new condition appearing in Theorem 3.1, is derived in §6, but it does not seem possible to replace this condition by an essentially simpler one. It is in fact shown (§7) that the result given in [11; Theorem 3.3] is the best that can be given in answer to the question raised in [11] in connection with [11; (3.12)]. If the set S, to which reference is made in that question, is more than countable, it becomes necessary to assume in the corresponding theorem some condition such as (3.5). On the other hand, it is shown (§5) that the set of conditions given in 3.1, though necessary for the validity of the conclusion (3.2) of 3.1, actually implies more than (3.2).

As observed in [11; §1], the problem under investigation can be considered to be an extension of a problem treated by Lebesgue in [8]. Lebesgue's treatment corresponds to the case when the functions $g_n(x)$ are absolutely continuous, that is to say, when we have

$$g_n(x) = \int_a^x G_n(t)\,dt \qquad (a \leq x \leq b,\ n = 1, 2, \cdots) \tag{1.1}$$

(the integrals being taken in the Lebesgue sense). If the restriction (1.1) is introduced in [11; Theorem 3.3] and 3.1, results are obtained which were not considered in [8]. In particular, if in addition to (1.1) it is also assumed that

$$G_n(x) \neq 0 \text{ almost everywhere in } I \qquad (n = 1, 2, \cdots), \tag{1.2}$$

then 3.1 provides a solution of Lebesgue's problem in the case when F coincides with the class of Riemann integrable functions (§8).

2. **Preliminary lemmas.**

2.1. *Suppose that $g(x)$ is in* **V**, *the class of functions of bounded variation in the closed interval $I = [a, b]$. Let $g^*(x)$ denote any one of the functions which satisfy the following relations*:

(2.1) *$g^*(x) = g(x)$ on M, the set consisting of the continuity points of $g(x)$ and of the points a, b;*

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