SEMI-NILPOTENT IDEALS

By Jakob Levitzki

I. **Introduction.** It has been proved by Hopkins [2], [4] that from the minimal condition for the right ideals of a ring S it follows that the sum N^* of all nilpotent two-sided ideals of S is nilpotent and the ring S/N^* is semi-simple. This theorem has been generalized by Asano [1] who assumes the minimal condition only for all two-sided ideals of S and for all right ideals of the ring S/P, where P denotes an arbitrary prime-ideal of S. In a previous paper [3] the author has proved that every ring contains a uniquely determined two-sided ideal N with the following properties: (1) N is semi-nilpotent (i.e., each sub-ring of N generated by a finite number of elements is nilpotent); (2) N contains all right and left semi-nilpotent ideals; (3) the ring S/N does not contain semi-nilpotent ideals other than zero. In the present note it is shown that if S is a ring with minimal condition for *two-sided semi-nilpotent* ideals, then the ideal N is nilpotent. This theorem yields as an immediate consequence a generalization of the result of Hopkins (Theorem 5) and also leads by a simple argument to a result which includes the theorem of Asano (Theorem 6).

II. **Notations and preliminary remarks.** A ring T will be called *semi-nilpotent* if each finite subset of T generates a nilpotent ring; otherwise T is called *semi-regular*. Every nilpotent ring is semi-nilpotent, and every semi-nilpotent ring is a nil-ring. In a previous note [3] the author has proved that the sum N of all two-sided semi-nilpotent ideals of a ring S is a semi-nilpotent two-sided ideal which contains also all one-sided semi-nilpotent ideals of the ring. The ideal N will be called the *radical* of S. The radical of S/N is always zero [3]. The sum N^* of all two-sided nilpotent ideals of the ring and the sum N^{**} of all two-sided nil-ideals will be called the *specialized radical* and the *generalized radical* respectively. In the case of an algebra the ideals N^*, N and N^{**} coincide. In general $N^* \subseteq N \subseteq N^{**}$. Accordingly, one may define the following three classes of semi-primary rings with a similar structure as the algebras (in short: A-rings):

(a) A ring S is called *an A-ring* if the ring S/N is semi-simple; here N denotes the radical of S.

(b) A ring S is called a *specialized A-ring* if the specialized radical N^* of S is nilpotent and the ring S/N^* is semi-simple.

(c) A ring S is called a *generalized A-ring* if the ring S/N^{**} is semi-simple; here N^{**} denotes the generalized radical of S.

Since the specialized radical of S/N is zero, it follows by a well-known theorem of E. Noether that we have

Received May 6, 1943.