# SOME REMARKS ON SUBHARMONIC FUNCTIONS

By Maxwell Reade

1. If $f(x, y)$ is continuous in a bounded domain $\mathfrak{D}$ (a bounded non-null connected open set), then a necessary and sufficient condition that $f(x, y)$ be subharmonic in $\mathfrak{D}$ is that one of the inequalities (all fundamental properties of subharmonic functions used in this paper are discussed in [6])

$$f(x, y) \leq M(f; x, y; r) \equiv \frac{1}{\pi r^2} \iint_{D(x,y;r)} f(x + \xi, y + \eta)\, d\xi\, d\eta, \tag{1}$$

$$f(x, y) \leq m(f; x, y; r) \equiv \frac{1}{2\pi r} \int_{C(x,y;r)} f(x + \xi, y + \eta)\, ds, \tag{2}$$

$$M(f; x, y; r) \leq m(f; x, y; r) \tag{3}$$

hold for each circular disc

$$D(x_0, y_0; r) : (x - x_0)^2 + (y - y_0)^2 = \xi^2 + \eta^2 \leq r^2$$

lying in $\mathfrak{D}$; here $C(x, y; r)$ denotes the boundary of $D(x, y; r)$. If $f(x, y)$ has continuous partial derivatives of the second order in $\mathfrak{D}$, then a necessary and sufficient condition that $f(x, y)$ be subharmonic in $\mathfrak{D}$ is that $\Delta f(x, y) \geq 0$ in $\mathfrak{D}$, where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Relative to (1) and (2) we have the operators

$$P(f; x, y) \equiv \overline{\lim_{r \to 0}} \frac{M(f; x, y; r) - f(x, y)}{\frac{1}{8}r^2}, \tag{4}$$

$$B(f; x, y) \equiv \overline{\lim_{r \to 0}} \frac{m(f; x, y; r) - f(x, y)}{\frac{1}{4}r^2}, \tag{5}$$

introduced by Privaloff [5] and Blaschke [2], respectively. The operators $P(f; x, y)$ and $B(f; x, y)$ have led to interesting characterizations of subharmonic functions; for example, if $f(x, y)$ is continuous in $\mathfrak{D}$, then a necessary and sufficient condition that $f(x, y)$ be subharmonic in $\mathfrak{D}$ is that $P(f; x, y) \geq 0$ in $\mathfrak{D}$ (or $B(f; x, y) \geq 0$ in $\mathfrak{D}$).

$P(f; x, y)$ and $B(f; x, y)$ are examples of generalized Laplacian operators (for a novel discussion of generalized Laplacians see [4]), some of which have proved

Received March 8, 1943.