TOPOLOGICAL GROUPS AS SUBGROUPS OF LINEAR TOPOLOGICAL SPACES

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The concepts of boundedness, convexity, and normability of linear topological spaces can be generalized for topological Abelian groups. These concepts can be used as conditions for a topological Abelian group to be a subgroup of a linear topological space or of a Banach space. Boundedness and convexity lead to theorems on normability analogous to one given by Kolmogoroff for linear topological spaces [4]. I am indebted to Professor Michal for his encouragement and suggestions.

1. **Topological groups and linear topological spaces.** By a topological group, we mean an abstract group which has a Hausdorff topology [2; 228–229, (A), (B), (C), (5)] with respect to which the group operations are continuous. A topological Abelian group is a topological group whose abstract group is Abelian. A linear topological space is a linear space [1; 26] which has a Hausdorff topology with respect to which the fundamental operations $x + y$ and ax are continuous. A rational linear topological space is a topological Abelian group T such that the product of rational numbers a and elements x of T is defined and continuous simultaneously in a and x and satisfies the postulates:

1. $a(x + y) = ax + ay$.
2. $(a + b)x = ax + bx$.
3. $a(bx) = (ab)x$.
4. $1 \cdot x = x$.

It can be shown that a linear topological space is a topological Abelian group and has the following properties (which hold for a rational linear topological space if a is rational):

I. If $ax = 0$, either $x = 0$ or $a = 0$.

II. If U is an open set and $a \neq 0$, then aU is an open set. (aU will mean the set of all ax, where $x \in U$; U^n, the set of all $x = u_1 + u_2 + \cdots + u_n$, where each u_i is in U.)

III. If U is a neighborhood of the origin, there is a neighborhood V of the origin such that $aV \subset U$ for all a satisfying $| a | \leq 1$.

2. **Convexity and boundedness.** Kolmogoroff [4] and Tychonoff [7] have called a neighborhood U of the origin of a linear topological space convex if $ax + (1 - a)y$ is in U for any elements x and y of U and real number a in the

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