ROSSER'S GENERALIZATION OF THE EUCLID ALGORITHM

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Rosser has recently presented [2] a generalization of the Euclid algorithm to sets of vectors in a space of several dimensions. The algorithm is a scheme for shortening the longer of a given set of vectors by subtracting integral multiples of the shorter. The process terminates in a set of vectors which cannot be further shortened by the algorithm. An examination of the terminal set reveals that it has certain minimal properties which may be adapted to meet hitherto intractable requirements imposed on the original set of vectors. The algorithm is applied in [2] to a variety of celebrated problems, yielding very satisfactory results with remarkable efficiency.

Our attention is here concentrated on the algorithm and the terminal set. Explicit lower bounds are found for the squared lengths of linear combinations of the terminal set, with necessary and sufficient conditions in each case that the lower bounds be attained. This information yields new proofs of the main results of [2], along with new results which complete those in [2] to an extent beyond which it does not appear possible to go in this particular direction. For example, in [2; Theorems 9–11 ff.] the effectiveness of the algorithm for its intended purpose is established under certain conditions, which are here weakened (Theorem 6) to the full extent permitted by the basic character of the problem.

Although the present paper is self-contained, the reader is referred to [2] for the setting of the problem, numerous important applications of the algorithm and a wealth of numerical illustrations. The author is grateful to Professor Rosser for granting him access to the manuscript of [2] before its appearance in print. In addition, Professor Rosser has kindly examined the present paper and has offered a great many valuable suggestions in the course of its preparation.

1. **Introduction. Definitions.** Let Y_i denote a finite set of vectors (located at the origin) in a Euclidean space of any number of dimensions. The vector $Z = \sum a_i Y_i$ is said to be an integral linear combination (ILC) of the Y's if and only if each a_i is an integer. The set Y_i is said to be commensurable if and only if the set of all ILC's of the Y's has no limit point. Throughout this paper, every set of vectors occurring is assumed to be commensurable. A set of *independent* vectors Z_i is said to be a greatest common factor (GCF) of the set Y_i if and only if each Z is an ILC of the Y's and conversely.

It is proved [2; Theorem 5 ff.] that a GCF can be exhibited for the set Y_i if and only if it is a commensurable set. If Z_i is a GCF of Y_i, and $Z'_i = \sum a_{ik} Z_k$, then it is easy to see that Z'_i will also be a GCF of Y_i if and only if the matrix

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