ANALYTIC FUNCTIONS IN CIRCULAR RINGS

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1. **Introduction.** Suppose $f(z)$ is meromorphic for $q \leq |z| \leq 1$, and that

$$|f(z)| \leq p \text{ for } |z| = q, \qquad |f(z)| \leq 1 \text{ for } |z| = 1.$$

(It is understood that $f(z)$ is single valued.) Suppose further that $f(z)$ has zeros at $a_1, a_2, \cdots, a_k$ at least, and poles at $b_1, b_2, \cdots, b_l$ at most, where the a's and b's are interior points of the ring. *If z_0 is a point within the ring, what is the largest possible value for $|f(z_0)|$?*

The multiplicities of the zeros and poles are to be considered. By saying that $f(z)$ has zeros at $a_1, \cdots, a_k$ at least, and poles at $b_1, \cdots, b_l$ at most, we mean that

$$f(z) \frac{(z - b_1) \cdots (z - b_l)}{(z - a_1) \cdots (z - a_k)}$$

is regular in the ring. (Likewise, if we say that $f(z)$ has zeros only at $a_1, \cdots, a_k$, and poles only at $b_1, \cdots, b_l$, we shall mean that the displayed function is regular and not zero in the ring.)

The problem raised in the first paragraph is solved in this paper. The condition that $f(z)$ is regular on the boundaries is not essential, but makes the formulation of the theorems somewhat simpler. Since any doubly connected region, neither of whose boundaries reduces to a point, can be mapped conformally on a circular ring, *we may regard the corresponding problem as solved for doubly connected regions in general.* On the other hand, the methods given apply to triply connected regions only in special cases.

In this introduction, we shall discuss three particular cases of our problem. The results may be regarded as extensions of (A) *the principle of maximum,* (B) *Hadamard's three circles theorem,* and (C) *Schwarz's lemma.*

(A) $p = 1, k = 0, l = 1, b_1 < 0, z_0 > 0$. In other words, $f(z)$ is regular for $q \leq |z| \leq 1$ except possibly for a simple pole at $-b$, where $q < b < 1$, and $|f(z)| \leq 1$ on the boundaries. The conclusion is that $|f(z)| \leq 1$ also for $q < z < 1$, i.e., for points directly opposite the pole. This result is proved in §3, and constitutes the *fundamental lemma* on which the entire paper is based.

According to the principle of maximum, if $f(z)$ is regular within and on the boundary of a region, and if $|f(z)| \leq 1$ on the boundary, then also $|f(z)| \leq 1$ within the region. The fundamental lemma is an extension of this principle, since it shows that in certain cases it remains valid even though $f(z)$ has a pole.

Note that no such extension of the principle of maximum is possible for simply connected regions. For example, if $f(z)$ is regular for $|z| \leq 1$ except for a simple

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