CONJUGATE HARMONIC FUNCTIONS

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We shall say that a set of real functions

$$x_j \equiv x_j(u_1, u_2, \cdots, u_n) \qquad (j = 1, 2, \cdots, m; m \geq n \geq 2),$$

defined and continuous in a domain (non-null connected open set) D, forms a *set of conjugate harmonic functions in D* provided

$$x_j(u_1, u_2, \cdots, u_n) \qquad (j = 1, 2, \cdots, m) \tag{1}$$

is harmonic in D;

$$\sum_{j=1}^{m} \frac{\partial x_j}{\partial u_k} \frac{\partial x_j}{\partial u_l} = [\lambda(u_1, u_2, \cdots, u_n)]\delta_{k,l} \qquad (k, l = 1, 2, \cdots, n), \tag{2}$$

where $\delta_{k,l}$ is the Kronecker delta defined by $\delta_{k,l} = 1$ or 0 as $k = l$ or $k \neq l$.

As indicated in (2), the function $\lambda(u_1, u_2, \cdots, u_n)$ is the same for all k, l; conditions (2), which in the theory of surfaces reduce to the familiar $E = G$, $F = 0$, are necessary and sufficient conditions that the functions $x_j(u_1, u_2, \cdots, u_n)$ map D conformally on an n-dimensional subspace of Euclidean m-space.

Given a real exponent γ, we define *the class S_γ* as the class of all functions $f(u_1, u_2, \cdots, u_n)$ which are continuous and non-negative in D and which are such that $f^\gamma \operatorname{sgn} \gamma$ is subharmonic if $\gamma \neq 0$, and $\log f$ is subharmonic if $\gamma = 0$. We note that a necessary and sufficient condition that a function f be of class S_γ is that it be of class S_β for all $\beta > \gamma$.

Similarly, we define *the class H_γ* as the class of all functions $f(u_1, u_2, \cdots, u_n)$ which are continuous and non-negative in D and which are such that in the part of D where $f > 0$, $f^\gamma \operatorname{sgn} \gamma$ is harmonic if $\gamma \neq 0$, and $\log f$ is harmonic if $\gamma = 0$. We note that the class H_γ is contained in the class S_γ.

The following result was obtained by Cioranesco [4].

Theorem 1. *A necessary and sufficient condition that the functions*

$$x_j \equiv x_j(u_1, u_2, \cdots, u_n) \qquad (j = 1, 2, \cdots, n; n \geq 2),$$

defined and continuous in a domain D, be a set of conjugate harmonic functions in D is that the following conditions be satisfied:

$$x_j(u_1, u_2, \cdots, u_n) \qquad (j = 1, 2, \cdots, n) \tag{3}$$

is harmonic in D;

Received January 18, 1943.