THE FIRST CANONICAL PENCIL

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1. **Introduction.** It is the purpose of this paper to give several geometric definitions for a general canonical line of the first kind. By generalizing a result of Sullivan [4; 117], we shall see how each such canonical line except the first axis of Cech may be defined in terms of the cusp-axes of the two families of hypergeodesics which are the extremals of the integrals

$$\int \beta^n \gamma^{1-n} v'^{2-3n}\, du, \qquad \int \gamma^n \beta^{1-n} v'^{3n-1}\, du, \tag{1.1}$$

where n is a constant. We also generalize a result of Rasmusen [5] to show that each canonical line except the axis of Cech may be regarded as the cusp-axis of a cone of class three which is defined by means of the osculating planes of the extremals of the integrals (1.1). The investigations of Sullivan and Rasmusen were for the case $n = 1$. In case $n = \frac{1}{2}$, the integrals (1.1) coincide and their extremals are the projective geodesics, the cusp-axis of which is the projective normal. By introducing at each point of the surface the triple of directions D_k^* which are conjugate to the directions D_k considered by Bell [1] in his investigations on the second canonical pencil, we arrive again at the general canonical line of the first kind as the intersection of the osculating planes of the projective geodesics in the directions D_k^*. (We have followed the terminology of Lane [4] in calling lines of the first kind those lines which Bell calls lines of the second kind. This accounts for the discrepancy in notation between this paper and his paper [1].) Finally, we apply our methods to a construction of Hsiung [2] to give still another determination of the general canonical line of the first kind.

2. **Analytic basis.** For the definitions and results to be stated in this section, see [4; Chapter III]. Let S be a non-ruled surface in ordinary projective space. If the asymptotic curves be chosen as parametric, then S is an integral surface of a pair of differential equations which can be reduced to the Fubini canonical differential equations

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v , \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v , \end{aligned} \tag{2.1}$$

where the subscripts indicate partial differentiation and the coefficients are functions of u, v such that $\beta\gamma \neq 0$ and $\theta = \log \beta\gamma$. The most general transformation of parameters and proportionality factor leaving the form of (2.1) invariant is

$$x = cx^*, \qquad u^* = U(u), \qquad v^* = V(v), \qquad cU'V' \neq 0, \tag{2.2}$$

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