DIFFERENTIABLE EVEN FUNCTIONS

By Hassler Whitney

An even function $f(x) = f(-x)$ (defined in a neighborhood of the origin) can be expressed as a function $g(x^2)$; $g(u)$ is determined for $u \geq 0$, but not for $u < 0$. We wish to show that g may be defined for $u < 0$ also, so that it has roughly half as many derivatives as f. A similar result for odd functions is given.

Theorem 1. *An even function $f(x)$ may be written as $g(x^2)$. If f is analytic, of class C^∞ or of class C^{2s}, g may be made analytic, of class C^∞ or of class C^s, respectively.*

If $f(x) = \sum a_i x^i$ is even and analytic, then $a_i = 0$ for i odd, and we may set $g(u) = \sum a_{2i} u^i$, which is analytic.

Suppose f is even and of class C^{2s}. Then Taylor's formula gives

$$f(x) = a_0 + a_1 x^2 + \cdots + a_{s-1} x^{2s-2} + x^{2s}\phi(x). \tag{1}$$

By Theorem 1 of [3], ϕ is even and continuous, and of class C^{2s} for $x \neq 0$, and

$$\lim_{x \to 0} x^k \phi^{(k)}(x) = 0 \qquad (k = 1, \cdots, 2s). \tag{2}$$

Set $\psi(u) = \psi(-u) = \phi(u^{\frac{1}{2}})$, and

$$g(u) = a_0 + a_1 u + \cdots + a_{s-1} u^{s-1} + u^s \psi(u). \tag{3}$$

Then $g(x^2) = f(x)$. To show that g is of class C^s, it is sufficient to show, by Theorem 2 of [3], that

$$\lim_{u \to 0} u^k \psi^{(k)}(u) \qquad (k = 0, \cdots, s) \tag{4}$$

exists.

If we differentiate $\psi(x^2) = \phi(x)$ $(x > 0)$, a simple proof by induction shows that, for some constants α_{ki},

$$\phi^{(k)}(x) = \sum_{1 \leq i \leq \frac{1}{2}k} \alpha_{ki} x^{k-2i} \psi^{(k-i)}(x^2) + 2^k x^k \psi^{(k)}(x^2). \tag{5}$$

Solving these equations in succession gives, for some β_{ki},

$$2^k x^k \psi^{(k)}(x^2) = \phi^{(k)}(x) + \sum_{1 \leq i \leq k-1} \beta_{ki} x^{-i} \phi^{(k-i)}(x). \tag{6}$$

Hence,

$$x^{2k} \psi^{(k)}(x^2) = \sum_{0 \leq i \leq k-1} \beta'_{ki} x^{k-i} \phi^{(k-i)}(x),$$

and (4) for $x > 0$ follows from (2). Since $\psi(-u) = \psi(u)$, the theorem is proved for this case.

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