DIFFERENTIABILITY OF THE REMAINDER TERM IN TAYLOR'S FORMULA

By Hassler Whitney

1. **Introduction.** Taylor's formula with exact remainder may be written

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{1}{n!}x^n f_n(x), \tag{1}$$

where

$$f_n(x) = \frac{n}{x^n}\int_0^x (x-t)^{n-1} f^{(n)}(t)\,dt \qquad (x \neq 0). \tag{2}$$

The main object of the paper (Theorem 1) is to show that if f is of class C^{n+p} (i.e., it has continuous derivatives through the order $n + p$), then f_n is of class C^p, but not necessarily of higher class. The condition that f be of class C^{n+p} may be expressed purely in terms of f_n (Theorem 2). A generalization is given to several dimensions (Theorem 3). The functions are always assumed defined in a neighborhood of the origin. The second theorem will be used in the following note; the results of both papers are essential in studying the singularities of certain mappings (see the next following paper).

2. **Properties of the remainder term.** We prove

Theorem 1. *Let* $1 \leq n \leq n + p$, *let* f *be of class* C^{n+p} *(in a region about the origin), and let* (1) *and* (2) *hold. Set* $f_n(0) = f^{(n)}(0)$. *Then* f_n *is of class* C^p, *and of class* C^{n+p} *for* $x \neq 0$. *Its derivatives are, for* $k = 0, \cdots, p$,

$$f_n^{(k)}(x) = \frac{n}{x^{n+k}}\int_0^x (x-t)^{n-1} t^k f^{(n+k)}(t)\,dt \qquad (x \neq 0), \tag{3}$$

$$f_n^{(k)}(0) = \frac{n!k!}{(n+k)!} f^{(n+k)}(0). \tag{4}$$

Furthermore,

$$\lim_{x \to 0} x^k f_n^{(p+k)}(x) = 0 \qquad (k = 1, \cdots, n). \tag{5}$$

See also Corollary 2 below.

We require two formulas, assuming ϕ differentiable.

$$\int_0^x t^{k+1}\phi'(t)\,dt = x^{k+1}\phi(x) - (k+1)\int_0^x t^k \phi(t)\,dt; \tag{6}$$

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