CLASSES OF RESTRICTED LIE ALGEBRAS OF CHARACTERISTIC p, II

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1. The class of algebras considered in this paper is obtained as follows: Let Φ be a field of characteristic p and let $\mathfrak{A} = \Phi(x_1, \cdots, x_m)$ be the commutative associative algebra with the basis $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, $0 \leq \alpha_i < p$, where $x_i^0 = 1$ and $x_i^p = \xi_i$ is in Φ. Let $\mathfrak{L} = \mathfrak{D}(\mathfrak{A})$ be the restricted Lie algebra of derivations of $\mathfrak{A}$, i.e., the set of transformations d of $\mathfrak{A}$ that satisfy

$$(x + y)d = xd + yd, \qquad (x\alpha)d = (xd)\alpha, \qquad (xy)d = x(yd) + (xd)y$$

for x, y in $\mathfrak{A}$ and α in Φ. The fundamental operations in $\mathfrak{L}$ are addition, scalar multiplication, commutation, $[d, e] = de - ed$, and p-exponentiation, d^p. (We shall show in §9 that our results are valid also when we drop the operation $d \rightarrow d^p$ and consider $\mathfrak{L}$ as a Lie algebra in the ordinary sense.) The case in which $\mathfrak{A}$ is a field has been considered by the author in a previous paper [3] and the algebra $\mathfrak{L}$ obtained when $m = 1$ and $\xi = 1$ is equivalent to one discovered by Witt and studied by Zassenhaus [8] and by Ho-Jui Chang [2]. We shall show that for any m and ξ_i, $\mathfrak{L}$ is normal simple unless $m = 1$, $p = 2$, and we obtain the derivation algebra of $\mathfrak{L}$. The automorphisms of $\mathfrak{L}$ and conditions that two algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be isomorphic are given for $p \geq 5$.

Since the x's generate $\mathfrak{A}$, any derivation d is determined by its effect on the x_i. Moreover, we may choose elements $y_1, \cdots, y_m$ arbitrarily in $\mathfrak{A}$ and obtain a derivation d such that $x_i d = y_i$, see [3; 217]. Thus, we have a 1-1 correspondence between the elements of $\mathfrak{L}$ and vectors $(y_1, \cdots, y_m)$, where y_i ranges over $\mathfrak{A}$. If $d \rightarrow (y_1, \cdots, y_m)$ and $c \rightarrow (z_1, \cdots, z_m)$, then $d + c \rightarrow (y_1 + z_1, \cdots, y_m + z_m)$ and $d\alpha \rightarrow (y_1\alpha, \cdots, y_m\alpha)$, α in Φ. Hence, the correspondence is linear and so the dimensionality of $\mathfrak{L}$ over Φ is mp^m. We note also that $[d, c] \rightarrow (w_i)$, where

$$w_i = \sum_k \left(\frac{\partial y_i}{\partial x_k} z_k - \frac{\partial z_i}{\partial x_k} y_k\right).$$

An explicit formula for the vector corresponding to d^p would be rather difficult to write and so we shall be content to note that the component y_{pi} of this vector is obtained by the recursion formula

$$y_{pi} = \sum_{k_1, \cdots, k_{p-1}} \left(\frac{\partial}{\partial x_{k_{p-1}}} \cdots \left(\frac{\partial}{\partial x_{k_2}} \left(\frac{\partial y_i}{\partial x_{k_1}} y_{k_1}\right) y_{k_2}\right) \cdots \right) y_{k_{p-1}}.$$

Received July 15, 1942; presented to the American Mathematical Society, April 3, 1942.
Part I appeared in [5]; we follow the notation and terminology of that paper.