TWO-TO-ONE MAPPINGS OF MANIFOLDS

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Introduction. An exactly two-to-one mapping is one for which each inverse image consists of exactly two points. This notion was introduced by O. G. Harrold [4] who showed that no such continuous mapping could be defined over an arc. J. H. Roberts [8] later showed that no two-to-one continuous mapping could be defined over a closed two-cell. Other results for this type of mapping have been obtained by O. G. Harrold [3], P. W. Gilbert [2], and Venable Martin and J. H. Roberts [6]. Mappings which are both two-to-one and interior have been studied by G. T. Whyburn [10].

The present paper is concerned with two-to-one mappings which are closed mappings. (A *closed mapping* is a continuous mapping which carries closed sets into closed sets.) The spaces over which these mappings are defined are n-manifolds, with or without boundary. In §1 it is shown that a two-to-one closed mapping defined over an n-manifold, with or without boundary, induces a "natural" homeomorphism of that manifold into itself. This result is applied in §2 to show that no two-to-one continuous mapping can be defined over a closed three-cell. The proof of this latter result is easily modified to give an alternate proof of the corresponding result for an arc or a closed two-cell. It is also shown in §2 that no two-to-one closed mapping can be defined over Euclidean n-space for $n = 1, 2, 3$.

Throughout this paper the letter M will be used to denote a metric space with differing properties, as specified in the various theorems. For the majority of the theorems, M will denote an n-manifold (absolute), or else an n-manifold with boundary. In the latter case the boundary consists of a countable number of mutually exclusive $(n - 1)$-manifolds whose sum is closed in M, and each of which is open in the sum. T will denote an exactly two-to-one closed mapping defined over M. (For compact M, the term *closed*, with reference to T, will be dropped since any continuous mapping of a compact space is a closed mapping.) The set of inverse images under T is an upper semi-continuous collection G filling M, and every element of G is a pair of points. For each $x \in M$, let $s(x)$ denote the point such that x, $s(x)$ is an element of the collection G. Then $T(x) = Ts(x)$. Let $f(x) = \rho(x, s(x))$, where ρ is the metric on M. When M is a non-compact manifold, we take for ρ a metric under which M is complete. Let K denote the set of all points $x \in M$ at which $f(x)$ is continuous, and let L denote the subset of K consisting of those points x such that $f(x)$ is continuous both at x and at $s(x)$. A point set C will be called *integral* if $s(C) = C$.

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