AFFINE GEOMETRY OF VECTOR SPACES OVER RINGS

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1. **Introduction.** The intuitive ideas involved in the "geometry" of the points with integer coördinates (lattice points) in the affine real plane and the subsets (lines) of such points which are in alignment are here abstracted to their essential content, in the spirit and with the methods of E. Artin's investigation on affine geometry [1]. The unusual character of such a "plane" geometry arises from the fact that parallel lines cannot be defined simply as lines which do not intersect. It is curious that parallelism may be introduced as an abstract equivalence relation with certain natural properties.

In §2, a geometry of points and lines is defined in a 2-space over a general type of ring and is shown to have certain geometric properties I-VII. These properties are then adopted as axioms for an abstract geometry of undefined points and lines in §3 and are proved by introduction of coördinates to completely characterize the 2-space geometry.

2. **Affine geometry in a 2-space over a ring.** Let $\mathfrak{R} = (0, 1, \alpha, \beta, \cdots)$ be a ring with unit ($1\alpha = \alpha = \alpha 1$) and no zero divisors ($\alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$). If $\alpha = \beta\gamma$ we say γ divides α. There will be no occasion to consider left divisors. Suppose further that every two elements α, β, not both zero, have a g.c.d., i.e., there is a δ which divides α, β, and which is divisible by every δ' dividing α, β.

A number of properties of such a ring, embodied in the following lemmas, will be needed.

LEMMA 1. *If $\delta_2\delta_1 = 1$, then $\delta_1\delta_2 = 1$ and δ_1, δ_2 are units.*

For $\delta_2(\delta_1\delta_2 - 1) = (\delta_2\delta_1)\delta_2 - \delta_2 = \delta_2 - \delta_2 = 0$. Since $\delta_2 \neq 0$, $\delta_1\delta_2 = 1$.

LEMMA 2. *If δ is a g.c.d. of α, β, then $\delta\gamma$ is a g.c.d. of $\alpha\gamma$, $\beta\gamma$.*

Let δ' be a g.c.d. of $\alpha\gamma$, $\beta\gamma$, and write $\alpha = \alpha_1\delta$, $\beta = \beta_1\delta$, $\alpha\gamma = \alpha_2\delta'$, $\beta\gamma = \beta_2\delta'$. Since $\alpha\gamma = \alpha_1\delta\gamma$, $\beta = \beta_1\delta\gamma$, $\delta\gamma$ divides δ' and $\delta' = \delta_1\delta\gamma$. Thus $\alpha\gamma = \alpha_2\delta_1\delta\gamma$, $\beta\gamma = \beta_2\delta_1\delta\gamma$, and therefore $\delta_1\delta$ divides α, β. Hence $\delta = \delta_2\delta_1\delta$, $\delta_2\delta_1 = 1$, and $\delta' = \omega\delta\gamma$, where $\omega = \delta_1$ is a unit. Every common divisor of $\alpha\gamma$, $\beta\gamma$ divides δ' and hence $\delta\gamma$.

LEMMA 3. *Relations $\alpha_1\delta = \alpha_2\delta'$, $\beta_1\delta = \beta_2\delta'$, where none of the pairs (α_1, β_1), (α_2, β_2), (δ, δ') is the zero pair $(0, 0)$, and where the former two pairs are relatively prime (g.c.d. $= 1$), imply $\alpha_2 = \alpha_1\omega$, $\beta_2 = \beta_1\omega$, where ω is a unit of $\mathfrak{R}$.*

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