THE GROWTH OF SOLUTIONS OF A DIFFERENTIAL EQUATION

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It has been shown by D. Caligo [1] that if $A(x)$ is continuous in $0 \leq x < \infty$ and $A(x) = O(x^{-2-\epsilon})$ as $x \to \infty$, with $\epsilon > 0$, then for any solution $y(x)$ of $y''(x) + A(x)y(x) = 0$, $\lim_{x\to\infty} y'(x)$ exists. It follows at once (e.g., by l'Hospital's rule) that $y(x)/x$ has the same limit.

Here we shall give a short proof of a somewhat more general result (Theorem 1), and then prove a still more extensive result (Theorem 2).

1. THEOREM 1. *If $A(x)$ and $B(x)$ are continuous in $0 \leq x < \infty$, if*

$$\int_0^\infty x \mid A(x) \mid dx < \infty, \tag{1.0}$$

and if

$$\int_0^{\to\infty} B(x)\, dx \tag{1.1}$$

exists, then for any solution $y(x)$ of

$$y''(x) + A(x)y(x) = B(x), \tag{1.2}$$

$$\lim_{x\to\infty} y'(x) \textit{ exists.} \tag{1.3}$$

In the proofs of Theorems 1 and 2 we require the following lemma.

LEMMA. *If $f(x)$ is continuous in $0 \leq x < \infty$, if $M(x)$ denotes the maximum of $\mid f(t) \mid$ in $0 \leq t \leq x$, and if for some positive numbers α and x_0*

$$\mid f(x) \mid \leq \alpha + \tfrac{1}{2}M(x) \qquad (x \geq x_0), \tag{1.4}$$

then $f(x)$ is bounded in $0 \leq x < \infty$.

For the proof, we suppose $f(x)$ unbounded, and let λ be a number larger than both 2α and $M(x_0)$. Let x_1 be the greatest lower bound of numbers x such that $\mid f(x) \mid \geq \lambda$; such numbers exist because $f(x)$ is unbounded. Since $f(x)$ is continuous, $f(x_1) = \lambda$; since $\mid f(x) \mid < \lambda$ for $x < x_1$, $M(x_1) = \lambda$; since $M(x_0) < \lambda$, $x_1 > x_0$. From (1.4) with $x = x_1$ we now have $\mid f(x) \mid \leq \alpha + \frac{1}{2}\lambda < \lambda$, which is a contradiction.

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