COMPLETELY MONOTONE FUNCTIONS IN PARTIALLY ORDERED SPACES

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1. The theorem on completely monotone functions.

Theorem (Hausdorff-Bernstein-Widder [4]). *If $T(\alpha)$ is defined in $0 < \alpha < \infty$ and if for each α and h $(0 < h < \infty)$ the relations*

$$(1) \qquad (-1)^n \Delta_h^n T(\alpha) \geq 0, \qquad n = 0, 1, 2, \cdots$$

hold, where $\Delta_h^0 T(\alpha) \equiv T(\alpha)$, $\Delta_h^1 T(\alpha) = T(\alpha + h) - T(\alpha)$ and $\Delta_h^{n+1} T = \Delta_h^1(\Delta_h^n T)$, then $T(\alpha)$ can be represented in the form

$$(2) \qquad T(\alpha) = \int_0^\infty e^{-\alpha t}\, dE(t)$$

and vice versa, where $E(t)$ is a function in $0 \leq t < \infty$ for which $E(0) = 0$, and $\Delta E(t) \geq 0$.

Also, the limits $E(+0)$ and $E(t \pm 0)$, $0 < t < \infty$, are uniquely determined.

It is the purpose of the present note to point out that the theorem remains true if the values of $T(\alpha)$ and $E(t)$ instead of being numbers are elements of a suitable vector space S and that the space S may be as general as the wording of the theorem will allow.

In order to be able to state assumption (1), it is sufficient to require that S be a commutative group of addition and that it be partially ordered by a relation $T \geq 0$ with the properties:

(i) $T \geq T$,
(ii) $T \geq U$, $U \geq T$ *imply* $T = U$,
(iii) $T \geq U$, $U \geq V$ *imply* $T \geq V$,
(iv) $T \geq U$ *implies* $T + V \geq U + V$ *for any* V.

It is customary to add the following property:

(v) *Given T and U there exists an element V such that $V \geq T$, $V \geq U$,*

but we emphasize that this property will not be needed.

Let us leave (1). However, as soon as we envisage the prospective relation (2), we are faced with the necessity of defining a Stieltjes integral of the type

$$\int_a^b \varphi(t)\, dE(t),$$

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