THE DECOMPOSITION OF MEASURES, II

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The main purpose of this paper is to prove that a flow on a measure space may be split into ergodic parts. The first result of this type is due to von Neumann,[1] who worked with metric, complete, separable spaces. For some important applications, particularly in probability theory, it is necesary to dispense with these topological assumptions. We shall make extensive use of the terminology, notation, and results of (D)[2] and (S).[3] Incidentally, we find it necessary to relax somewhat the strict separability conditions of (D), redefine "direct sum", and prove a generalization of the general decomposition theorem of (D). As for the decomposition of a flow, we have chosen to reduce this to the case of a single measure preserving transformation by means of Theorem 4 of (S).

A measure space $\Omega(\mathfrak{M}, m)$ is *properly separable* if there exists a strictly separable Borel field $\mathfrak{B} \subseteq \mathfrak{M}$, such that for every $M \in \mathfrak{M}$ there is a $B \in \mathfrak{B}$ with $M \subseteq B$ and $m(B - M) = 0$. Throughout this paper we assume that all measure spaces considered are properly separable and complete, in the sense that any subset of a measurable set of measure zero is itself measurable. Ω is said to be a *direct sum* of the measure spaces $Y_x(\mathfrak{Y}_x, \nu_x)$ formed with respect to the measure space $X(\mathfrak{X}, \mu)$, in symbols

$$\Omega(\mathfrak{M}, m) = \int_{X(\mathfrak{X}, \mu)} Y_x(\mathfrak{Y}_x, \nu_x)\, d\mu(x),$$

if the conditions of §3 in (D) are satisfied, with the exception that we require only that for every $M \in \mathfrak{M}$, MY_x be a measurable subset of Y_x for *almost every* x.

Theorem 1. *If $\Omega(\mathfrak{M}, m)$ is a measure space and $\mathfrak{A}$ a Borel field, $\mathfrak{A} \subseteq \mathfrak{M}$, then there exists a set $A \in \mathfrak{A}$ of measure zero such that $\Omega - A$ is a direct sum,*

$$\Omega - A = \int_X Y_x\, d\mu(x),$$

in such a way that the Borel field $\mathfrak{X}$ of all measurable x-sets is contained in and is equivalent to the given Borel field $\mathfrak{A}$.

Proof. Let $\mathfrak{B}$ be a strictly separable Borel field related to $\mathfrak{M}$ as in the definition of proper separability, and let $\mathfrak{A}'$ be any strictly separable Borel field con-

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[1] J. v. Neumann, *Zur Operatorenmethode in der klassischen Mechanik*, Annals of Mathematics, vol. 33(1932), pp. 587–642. See p. 617.

[2] (D): Paul R. Halmos, *The decomposition of measures*, Duke Mathematical Journal, vol. 8(1941), pp. 386–392.

[3] (S): W. Ambrose and S. Kakutani, *Structure and continuity of measurable flows*, Duke Mathematical Journal, vol. 9(1942), pp. 25–42.