STRUCTURE AND CONTINUITY OF MEASURABLE FLOWS

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1. **Introduction.** The purpose of this paper is to establish some regularity properties for flows which are assumed to satisfy only measurability conditions. In particular, we are concerned with establishing conditions under which a flow will be isomorphic to a continuous flow or a flow built under a function. For measure spaces which satisfy the two conditions of being properly separable and having a separating sequence of measurable sets (see Definitions 8 and 10),[1] our results can be summed up by saying that every flow different from the identity is isomorphic to a (generalized) flow built under a function and to a continuous flow on a separable metric space with a regular measure; these results are obtained in Theorems 2, 4 and 5. For measure spaces which do not satisfy these conditions, the situation is more complicated, and we refer the reader to the body of the paper. Flows built under a function were first introduced, in a special case, by J. von Neumann[2] and have since been considered by one of the present authors.[3] The significance of this isomorphism of any flow to such a flow is that it gives a kind of normal form for a flow and it makes possible the taking of cross sections and the reduction of various properties of a flow to properties of a single measure preserving transformation on such a cross section.

2. **Definitions and notation.**

DEFINITION 1. A *measure space* $\Omega(\mathfrak{B}, m)$ is a system of a space Ω, a Borel field[4] $\mathfrak{B}$ of subsets M of Ω, and a countably additive measure[5] $m(M)$ defined on $\mathfrak{B}$ and satisfying the following conditions:

(2.1) $\Omega \epsilon \mathfrak{B}$ and $m(\Omega) < +\infty$,

(2.2) there exists an $M \epsilon \mathfrak{B}$ for which $0 < m(M) < m(\Omega)$,

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[1] All measures usually considered, and in particular Lebesgue measures in Euclidean spaces, satisfy these conditions.

[2] J. von Neumann [4], pp. 636–641.

[3] W. Ambrose [1].

[4] A collection of subsets of a space is called a *field* if it is closed under the operations of finite addition, finite intersection and complementation. It is a *Borel field* if it is a field and is closed under the operations of countable addition and intersection. It is easy to see that for any collection of subsets of a space there exists a smallest field, and also a smallest Borel field, which contains the given collection; these are called respectively the field *determined* by the collection and the Borel field *determined* by the collection.

[5] A *countably additive measure* is a non-negative set function $m(M)$ defined for all sets M in some Borel field $\mathfrak{B}$ and having the property that $m\left(\sum_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} m(M_n)$ for any sequence $\{M_n\}$ $(n = 1, 2, \cdots)$ of disjoint sets from $\mathfrak{B}$.