THE CANONICAL LINES

By Chuan-Chih Hsiung

1. **Introduction.** Associated with a point P of a surface S several covariant lines have been defined by various authors in independent investigations of the projective differential geometry of surfaces. Among them perhaps the most important are the directrices of Wilczynski,[1] the axes of Čech,[2] the edges of Green,[3] and the projective normal of Green and Fubini.[4] In view of the fact that all of these lines just mentioned that pass through the point P are characterized by apparently unrelated properties, it has been considered remarkable that they all should lie in a plane. This plane is called the *canonical plane.* Any line passing through the point P and lying in the canonical plane is spoken of as a *canonical line of the first kind.* The reciprocal polar lines of the canonical lines of the first kind with respect to the quadric of Lie or any quadric of the pencil of Darboux at the point P of the surface S, dually, lie in the tangent plane of S at P and pass through a common point, which is called the *canonical point.* Any of these lines is spoken of as a *canonical line of the second kind.*

The purpose of this note is to present a new geometric characterization for a general canonical line of each kind, and especially for the first axis of Čech.

2. **Analytic basis.** In ordinary space in which a point has projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$, the parametric vector equation of an analytic non-ruled surface S is

$$x = x(u, v), \tag{1}$$

the parameters being u, v. If the asymptotic curves on the surface S are the parametric curves, the coördinates x satisfy a system of two partial differential equations which can be reduced to *Fubini's canonical form*

$$\begin{cases} x_{uu} = px + \theta_u x_u + \beta x_v\,, \\ x_{vv} = qx + \gamma x_u + \theta_v x_v \end{cases} \qquad (\theta = \log \beta\gamma), \tag{2}$$

subscripts indicating partial differentiation and the coefficients being functions of u, v which satisfy certain integrability conditions. Then the coördinates of

Received March 4, 1941.

[1] E. J. Wilczynski, *Projective differential geometry of curved surfaces* (Memoirs 2–3), Transactions of the American Mathematical Society, vol. 9(1908), pp. 79–120; 293–315.

[2] E. Čech, *L'intorno di un punto d'una superficie considerato dal punto di vista proiettivo,* Annali di Matematica Pura ed Applicata, (3), vol. 31(1922), pp. 191–206.

[3] G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences,* Transactions of the American Mathematical Society, vol. 20(1919), pp. 79–153.

[4] G. Fubini, *Fondamenti della geometria proiettivo-differenziale di una superficie,* Reale Accademia delle Scienze, Torino, Atti, vol. 53(1918), pp. 1032–1043. See also G. M. Green, Bulletin of the American Mathematical Society, vol. 23(1916), pp. 73–74, Abstract.