THE SUM OF THE DIVISORS OF A POLYNOMIAL

By E. F. Canaday

1. Introduction. Let

$$A = A(x) = x^k + \alpha x^{k-1} + \cdots + \lambda$$

denote a polynomial with coefficients modulo 2. All coefficients may then be written 1 or 0 and the number of polynomials of degree k is 2^k.

Let $\sigma(A)$ denote the sum of the divisors of A. If $\sigma(A_1) = A_2$, we write $A_1 \to A_2$. Clearly A_1 and A_2 are of equal degree. Consider the sequence $A_1 \to A_2 \to A_3 \to \cdots$ where $\sigma(A_j) = A_{j+1}$. Since the number of polynomials of any given degree is finite, after a certain point an A_i in the sequence will be repeated. If $A_1 \to A_2 \to \cdots \to A_n \to A_1$, all A_i being distinct, the set $A_1, \cdots, A_n$ will be called an n-ring. In particular, if $A \to A$, i.e. $\sigma(A) = A$, we shall call A a one-ring. Other definitions and notation are given in section 2.

Section 3 contains theorems on weight. In section 4 are developed some invariant properties of the operator σ. Section 5 consists principally of lemmas needed for subsequent theorems. It also contains the useful theorem: *The only complete polynomials whose irreducible factors are all of the form* $x^\alpha(x+1)^\beta + 1$ *are* $x^2 + x + 1$, $x^4 + x^3 + x^2 + x + 1$ *and* $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

In sections 6–8, one-rings and methods of constructing them are discussed. First the trivial type $x^{2^n-1}(x+1)^{2^n-1}$ is treated, then the type $x^h(x+1)^k A$, of which eleven are found, and then a proof is given that there are no others of certain sub-types. Lastly the type B^2, where $(B, x(x+1)) = 1$, is discussed but none found. It seems plausible that none of this type exist but this is not proved.

Section 9 is devoted mainly to two-rings. There is the infinite class

$$x^{2^\alpha-1}(x+1)^{2^\beta-1} \leftrightarrow x^{2^\beta-1}(x+1)^{2^\alpha-1} \qquad (\alpha \neq \beta)$$

corresponding to the infinite class of trivial one-rings. In addition we determine the two-rings of certain forms. The simplest of these is

$$(1.1) \qquad x^\alpha(x+1)^\beta \to A \to x^\alpha(x+1)^\beta.$$

We show that there are only three of these.

Generalizing (1.1) we seek all rings of the form

$$(1.2) \quad x^{\alpha_1}(x+1)^{\beta_1} \to A_1 \to x^{\alpha_2}(x+1)^{\beta_2} \to A_2 \to \cdots \to A_P \to x^{\alpha_1}(x+1)^{\beta_1},$$

where alternate polynomials are of the form $x^\alpha(x+1)^\beta$. We show that there are only nine of this form, three two-rings (1.1) and six four-rings.

Received September 20, 1941.