A PARTITION FUNCTION CONNECTED WITH THE MODULUS FIVE

By Joseph Lehner

1. The purpose of this paper is to derive a convergent series for $p_1(n)$ and $p_2(n)$, the number of partitions of a positive integer n into summands of the form $5l \pm 1$ and $5l \pm 2$, respectively. These partition functions occur in the following theorems of I. Schur [7],[1] which can be regarded as further cases of a result due to Euler:[2]

A. *The number of partitions of n into summands whose minimal difference is two is equal to* $p_1(n)$;

B. *The number of partitions of n into summands whose minimal difference is two and in which the summand one does not occur is equal to* $p_2(n)$.

It is convenient to treat $p_1(n)$ and $p_2(n)$ together. The other possible case for the modulus 5, that in which all summands are divisible by 5, reduces trivially to the *unrestricted* partition function $p(n): p_0(n) = p(n/5)$. $p(n)$, on the other hand, has been fully discussed [2].

We follow the method of Rademacher [3]. Consider the generating functions

$$
\begin{aligned}
F_a(x) &= \prod_{m=0}^{\infty} (1 - x^{5m+a})^{-1} \prod_{m=1}^{\infty} (1 - x^{5m-a})^{-1} \\
&= 1 + \sum_{n=1}^{\infty} p_a(n) x^n \qquad (a = 1, 2),
\end{aligned} \tag{1.1}
$$

which converge inside the unit circle. In order to determine the asymptotic behavior of $F_a(x)$ near a "rational point" on a circle concentric to the unit circle but interior to it, we subject x to the transformation[3] $x \to x'$, where

$$
x = \exp\left[2\pi i \frac{h + iz}{k}\right], \qquad x' = \exp\left[2\pi i \frac{h' + iz^{-1}}{k}\right]. \tag{1.2}
$$

Here $\Re z > 0$, h and k are coprime integers satisfying $0 \leqq h < k$, and h' is any fixed solution of

$$
hh' \equiv -1 \pmod{k}. \tag{1.21}
$$

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[1] Numbers in square brackets refer to the bibliography at the end of this paper.

[2] Namely, the number of partitions of n into unequal parts (i.e., parts whose minimal difference is one) is equal to the number of partitions of n into odd parts (parts congruent ± 1 modulo 2).

[3] This amounts to a modular transformation of $F_a(x)$ considered as a function of τ: $x = \exp(2\pi i \tau)$. See §7.