YOUNG'S SEMI-NORMAL REPRESENTATION OF THE SYMMETRIC GROUP

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Introduction. The main purpose of this note is to give a new (shorter and more elementary) derivation of A. Young's semi-normal representation of the symmetric group. As a starting point we take the discussions by H. Weyl ([6], Chap. IV, §2, 3) and D. E. Littlewood ([4], Chap. V, especially §4).

Denote the partition $m = \lambda_1 + \cdots + \lambda_\kappa$, $\lambda_1 \geqq \cdots \geqq \lambda_\kappa > 0$ by (λ). We represent (λ) geometrically by an array of squares; λ_1 in the 1st row, $\cdots$, λ_κ in the κ-th row; the j-th squares of the rows making a column. The m squares or *fields* of the array are labelled by the numbers from 1 to m in such a way that the labels in every row increase from left to right and in every column increase from top to bottom. The array thus labelled is called a *regular Young diagram* belonging to the partition (λ).

Associated with each partition (λ) of m there is an irreducible matrix representation of the symmetric group, $\mathfrak{S}_m$, of degree m. The degree[1] $g(\lambda)$ of this representation is equal to the number of regular Young diagrams belonging to (λ). Let the label of the field in the α-th row and β-th column of a regular diagram, T, be denoted by $a(\alpha, \beta)$. If T and T' both belong to (λ) we say that

(1) T *precedes* T' if each of the fields labelled m, $m - 1, \cdots, m - r + 1$ lies in the same row in both diagrams, but the field $m - r$ lies in a lower row in T than in T'.

We enumerate the regular diagrams belonging to (λ) according to this ordering. Now number the partitions (λ) of m according to their dictionary order[2] and denote by $T(ij)$ the j-th regular Young diagram belonging to the i-th partition of m.

Corresponding to each diagram $T(ij)$ we shall define a primitive idempotent $e(ij)$ in the group $\mathfrak{K}$-ring, $\mathfrak{R}_m$, of $\mathfrak{S}_m$. [$\mathfrak{K}$ is here the field of complex numbers.]

Let $\epsilon(i) = \sum e(ij)$, summed for j from 1 to $g(\lambda^i)$. Then the two sided ideal $\epsilon(i)\mathfrak{R}_m$ of $\mathfrak{R}_m$ is a total matrix algebra $\mathfrak{A}_i = \mathfrak{A}(\lambda^i)$, of degree $g(\lambda^i)$, homomorphic with $\mathfrak{R}_m$ under the mapping $x \to x(i) = \epsilon(i)x$; and $\mathfrak{R}_m$ is the direct sum of the simple algebras $\mathfrak{A}_i$.

The next step is the choice of elements $e(ijk)$, $j, k = 1, \cdots, g(\lambda^i)$, which constitute an ordinary matrix basis ([1], p. 7) for $\mathfrak{A}_i$. In the terminology of representation theory the element x of $\mathfrak{R}_m$ is ordered to the matrix $B_i(x) =$

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[1] [4], Th. I, p. 68, Th. IV, p. 75; [6], Th. 7.7B, p. 213.

[2] That is, (λ) has a smaller number than (λ') if the first non-vanishing difference $\lambda_1 - \lambda_1'$, $\lambda_2 - \lambda_2'$, ... is positive.