A THEOREM ON DIMENSION

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The present paper grew from an attempt to answer a question of Hurewicz.[1] The answer is given by Theorem 9.1. It is also thought that Theorem 1.1 is of some importance. One application of it, other than Theorem 9.1, is given.

The letter R with positive integral subscript will denote Euclidean space of the indicated dimension.

1. THEOREM 1.1. *If the separable metric space M has dimension n, then there is a homeomorphism f of M into a subset of R_{2n+1} such that for any $R_k \subset R_{2n+1}$ $(n + 1 \leqq k \leqq 2n + 1)$ we have* $\dim (f(M) \cdot R_k) \leqq k - n - 1$.

This theorem is shown to be a consequence of the following special case:

THEOREM 1.2. *If the separable metric space M has dimension n, then there is a homeomorphism f of M into a subset of R_{2n+1} such that for any $R_{n+1} \subset R_{2n+1}$ we have* $\dim (f(M) \cdot R_{n+1}) \leqq 0$.

We show that the f satisfying the conclusion of Theorem 1.2 will do for Theorem 1.1. The proof is by induction on k. By hypothesis the result is true for $k = n + 1$. Suppose it is true for $k = r < 2n + 1$. Take any R_{n+1} in R_{2n+1}. Select $p \in f(M) \cdot R_{r+1}$. For arbitrary $\epsilon > 0$ there is a rectangular ϵ-domain U of p in R_{r+1} bounded by parts of a finite number of r-dimensional spaces $R_r^1, R_r^2, \cdots, R_r^i$. By our inductive hypothesis, $\dim (f(M) \cdot R_r^j) \leqq r - n - 1$ $(j = 1, 2, \cdots, i)$. Thus $\dim (f(M) \cdot R_{r+1}) \leqq (r - n - 1) + 1 = (r + 1) - n - 1$. This means that our conclusion holds for $k = r + 1$, and the argument is complete.

2. **Proof of Theorem 1.2.** We first imbed[2] M topologically in a compact space of dimension n. However, to avoid extra terminology we shall continue to use the letter M, which throughout the remainder of this proof will denote a compact n-dimensional metric space.

2.1. DEFINITION OF F. Let F denote the space of all continuous mappings of M into subsets of R_{2n+1} with the metric $| f_1 - f_2 | = \max_{x \in M} | f_1(x) - f_2(x) |$.

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[1] W. Hurewicz, *Ueber stetige Bilder von Punktmengen*, Proc. Amsterdam Academy, vol. 30(1927), p. 161, footnote 7. "Es entsteht die Frage ob bei vorgegebenen n^* und n $(n^* > n \geqq 0)$ sich jede n^*-dimensionale separable Menge als ein eindeutiges beiderseits stetiges Bild einer n-dimensionalen Menge mit höchstens $(n^* - n + 1)$-fachen Punkten darstellen lässt."

[2] This is possible, as proved by Hurewicz: *Ueber Einbettung separabler Räume in gleichdimensionale kompakte Räume*, Monatshefte für Math. und Physik, vol. 37(1930), pp. 199–208.