CONVERGENCE AND DIVERGENCE OF NON-HARMONIC GAP SERIES

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1. **Introduction.** The theory of convergence of trigonometric (harmonic) series with gaps of Hadamard's type is by now a closed chapter. However, new difficulties present themselves when one tries to get a parallel theory for series

$$(1) \qquad \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \lambda_k > 0; \lambda_{k+1}/\lambda_k > q > 1 \qquad (k = 1, 2, \cdots),$$

without assuming that the λ's are integers. The source of these difficulties is the fact that the non-harmonic series must necessarily be considered on the infinite interval $-\infty < t < \infty$. In the harmonic case the terms have the same period and the problem is simplified both by having to deal with a finite interval and by the properties of orthogonality. To meet the new situation methods of proof had to be modified and some new devices invented. It should, however, be mentioned that the solution obtained is not complete. In §3 we prove that if $\sum | a_k |^2 = \infty$ and $\lambda_{k+1}/\lambda_k > q > (5^{\frac{1}{2}} + 1)/2$ the series (1) diverges almost everywhere, whereas in the harmonic case it suffices to assume that $\lambda_{k+1}/\lambda_k > q > 1$. Thus far we have not been able to remove the condition involving $(5^{\frac{1}{2}} + 1)/2$.

2. **Convergence of non-harmonic gap series.**

Theorem 1. *If* $\sum | a_k |^2$ *converges, the series* (1) *converges almost everywhere for* $-\infty < t < \infty$.

The method of proof is a modification of a method of Marcinkiewicz.[1] Let $0 < \delta < \lambda_2 - \lambda_1$. Since $\lambda_{k+1} - \lambda_k \geqq \lambda_2 - \lambda_1$ $(k \geqq 1)$, the intervals $(\lambda_k - \delta, \lambda_k)$ do not overlap and we can define a function $f(x)$ by putting

$$f(x) = a_k \qquad \text{for } \lambda_k - \delta < x \leqq \lambda_k,$$
$$f(x) = 0 \qquad \text{otherwise.}$$

Obviously $f(x) \in L^2(-\infty, \infty)$ and hence its Fourier transform is $(C, 1)$ summable almost everywhere.[2] In particular, it follows that

$$(2) \qquad \lim_{n \to \infty} \int_0^{\lambda_n} \left(1 - \frac{x}{\lambda_n}\right) f(x) e^{itx}\, dx$$

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[1] J. Marcinkiewicz, *A new proof of a theorem on Fourier series*, Journal of the London Mathematical Society, vol. 8(1933), p. 279.

[2] See, for instance, E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937, pp. 84–85. The theorem in question is due to Plancherel.