RIEMANN SUMS AND THE FUNDAMENTAL POLYNOMIALS OF LAGRANGE INTERPOLATION

By J. H. Curtiss

1. **Introduction.** Let C denote an arbitrary Jordan curve of the complex z-plane, and let $z = \varphi(w)$ be an analytic function which maps the exterior K of C (i.e., the unlimited region bounded by C) conformally onto the region $|w| > 1$ so that the points at infinity correspond. We assume that this function is defined so as to be continuous and univalent for $1 \leqq |w| < \infty$. The Laurent series for the function can be written as follows:

$$\varphi(w) = cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, \qquad |w| \geqq 1, \tag{1.1}$$

where $|c|$ is the transfinite diameter of C.

The polynomials

$$\omega_n(z) = \prod_{k=1}^{n} [z - \varphi(e^{2\pi ik/n})] \qquad (n = 1, 2, \cdots)$$

are called the fundamental polynomials of Lagrange interpolation in the points $\varphi(e^{2\pi ik/n})$ on C. It is well known[1] that

$$\lim_{n\to\infty} |\omega_n(z)|^{1/n} = \begin{cases} |c|\,|w|, & z = \varphi(w),\ z \text{ in } K, \\ |c|, & z \text{ interior to } C. \end{cases}$$

In the present paper we attack the more delicate problem of determining the exact behavior of the sequence $\{\omega_n(z)\}$, rather than that of the sequence $\{|\omega_n(z)|^{1/n}\}$. The results to be established may be stated formally as follows:

Theorem 1. *If C is rectifiable, then*

$$\lim_{n\to\infty} \frac{\omega_n(z)}{-c^n} = 1 \tag{1.2}$$

uniformly for z on any closed set M of the region interior to C, and

$$\lim_{n\to\infty} \frac{\omega_n(z)}{c^n(w^n - 1)} = 1, \qquad z = \varphi(w), \tag{1.3}$$

uniformly for z on any closed set M_1 of K.

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[1] L. Fejér, *Interpolation und konforme Abbildung*, Göttinger Nachrichten, 1918, pp. 319–331; J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20, New York, 1935; especially pp. 68–75.