INTRANSITIVE ABELIAN ALMOST-TRANSLATION GROUPS OF ALMOST-PERIODIC FUNCTIONS

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It is our purpose here to generalize as completely as possible the results which H. Bohr and D. A. Flanders obtained in a recent paper[1] concerning the almost-translation groups of almost-periodic functions.

We deal with a finite set of m distinct one-valued, continuous almost-periodic functions $f_1(t), \cdots, f_m(t)$ (written collectively as $[f(t)]$) of a real variable, defined for $-\infty < t < +\infty$. The substitution S is defined over the integers $1, \cdots, m$, and if S takes j into k, then $S\{f_j(t)\} = f_k(t)$. For given $\epsilon > 0$ the real number τ is said to ϵ-perform the substitution S on $[f(t)]$ if

$$[| f_h(t+\tau) - Sf_h(t) | < \epsilon] \qquad (-\infty < t < +\infty;\ h = 1, \cdots, m).$$

A substitution S on $[f(t)]$ is defined as an *almost-translation substitution* if the set $\{\tau_{(S)}(\epsilon)\}$ of all real numbers τ each of which ϵ-performs the substitution S on $[f(t)]$ is relatively dense for every positive ϵ.

The set of almost-translation substitutions of $[f(t)]$ forms an Abelian group called the *almost-translation group* G of $[f(t)]$. Bohr and Flanders set up a list of necessary and sufficient conditions which a set $[f(t)]$ must satisfy in order that it have a given transitive Abelian group G as its almost-translation group, and then they went on to prove that any transitive group G can serve as the almost-translation group of some set $[f(t)]$. The authors left open the question of intransitive Abelian groups. We settle this question conclusively and arrive at results which cannot be generalized further since almost-translation groups must be Abelian substitution groups. We denote the main theorem of Bohr and Flanders by Theorem A and present it with changes in notation.

Theorem A.[2] *Let $[f(t)]$ be a finite set of m almost-periodic functions, $_1(t), \cdots, f_m(t)$; and let G be an arbitrary transitive Abelian group of m substitu-ions which we denote by*

$$S(1) = \begin{pmatrix} 1, 2, \cdots, m \\ 1, 2, \cdots, m \end{pmatrix}, \cdots, S(m) = \begin{pmatrix} 1, 2, \cdots, m \\ m, \cdots \end{pmatrix}.$$

Then in order that $[f(t)]$ be composed of distinct functions and have G as its almost-translation group, it is necessary and sufficient that the following four conditions be fulfilled:

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[1] *Algebraic equations with almost-periodic coefficients*, Kgl. Danske Videnskabernes Selskab, Mathematisk-fysiske Meddelelser, vol. 15(1937), pp. 1–49.

[2] Loc. cit., p. 32.