FUNCTIONS HAVING SUBHARMONIC LOGARITHMS

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1. **Introduction.** A function $x(u, v)$, defined in a domain D (non-null connected open set), such that $x(u, v) < +\infty$, is said to be *subharmonic*[1] in D provided it satisfies the following conditions:

1.1. $x(u, v)$ is upper semi-continuous in D;

1.2. for every domain D' comprised together with its boundary B' in D, and for every function $h(u, v)$, harmonic in D', continuous in $D' + B'$, and satisfying $h(u, v) \geqq x(u, v)$ on B', we have also $h(u, v) \geqq x(u, v)$ in D'.

A function $x(u, v)$, which is upper semi-continuous and $\not\equiv -\infty$ in D, is subharmonic there if and only if it satisfies

$$x(u_0, v_0) \leqq \frac{1}{2\pi} \int_0^{2\pi} x(u_0 + \rho \cos \varphi, v_0 + \rho \sin \varphi)\, d\varphi$$

for every point (u_0, v_0) in D and for every ρ such that the circular disc $(u - u_0)^2 + (v - v_0)^2 \leqq \rho^2$ is comprised in D. A function $x(u, v)$, having continuous second derivatives in D, is subharmonic there if and only if its Laplacian satisfies the inequality

$$\Delta x(u, v) \equiv \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} \geqq 0.$$

A function $p(u, v)$, defined in a domain D, is said to be[2] *of class PL* in D provided $p(u, v) \geqq 0$ and $\log p(u, v)$ is subharmonic there.

A function of class PL necessarily is subharmonic, but the converse is by no means true. Indeed, a non-negative function $p(u, v)$ is of class PL if and only if $[p(u, v)]^\alpha$ is subharmonic for all choices of the positive constant α. Other criteria for functions of class PL are contained in the following Theorems A and B.

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[1] For a discussion of subharmonic functions, see T. Radó, *Subharmonic Functions*, Berlin, 1937. There, as usual, subharmonic functions are defined to be functions which satisfy $x(u, v) \not\equiv -\infty$, in addition to the conditions we have here given; but, for example, $\log | f(u + iv) |$, where $f(z)$ is analytic, is subharmonic, and we prefer not to exclude the important special case $f(z) \equiv 0$. The discussions in the present paper would not, however, be affected by the restriction $x(u, v) \not\equiv -\infty$.

[2] See E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces* and *Subharmonic functions and surfaces of negative curvature*, Transactions of the American Mathematical Society, vol. 35(1933), pp. 648–661 and 662–674 for definition and applications of functions of class PL.