THE DECOMPOSITION OF MEASURES

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1. Introduction. The main purpose of this paper is to prove a theorem on the decomposition of measures (Theorem 1 of §5)—a theorem which asserts that under certain hypotheses a measure space can be expressed as a direct sum of measure spaces. Although the result is of a certain independent interest, it is proposed chiefly as a new method: a tool to be used in the theory of measure-preserving transformations. In §6 we shall give, using this method, an easy proof of a theorem, due to von Neumann, on the decomposition of an arbitrary measure-preserving transformation into ergodic parts. In a subsequent paper we shall apply the method to the spectral theory of measure-preserving transformations to obtain a decomposition of an arbitrary measure-preserving transformation into parts which have either pure point spectrum or pure continuous spectrum.

2. Measure spaces and separability. Let Ω be any set of elements ω and let $\mathfrak{B}$ be a Borel field of subsets of Ω.[1] We suppose that on $\mathfrak{B}$ there is defined a measure m with $m(\Omega) = 1$: m is a non-negative, countably additive function of sets. We shall call Ω, together with $\mathfrak{B}$ and m, a *measure space*; when necessary we shall write $\Omega(\mathfrak{B}, m)$ for Ω, to emphasize the particular Borel field and measure under consideration. All the Borel fields we shall consider will be supposed to be (not necessarily proper) subfields of $\mathfrak{B}$, so that we may assume that the measure m is defined on them. If $\mathfrak{A}$ is a Borel field and A a set, $A \epsilon \mathfrak{A}$, we shall say that A is measurable ($\mathfrak{A}$); instead of measurable ($\mathfrak{B}$) we shall generally say measurable. A similar terminology will be used concerning the measurability of functions. We shall call the smallest Borel field containing a given collection of sets the Borel field spanned by them. For two sets, functions, transformations, etc., we shall use the symbol $\doteq$ to denote the fact that they are equal except possibly for a set of measure zero (i.e., equal *almost everywhere* or *a. e.*).[2] Two Borel fields $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (both contained in $\mathfrak{B}$) will be called *equivalent*, in symbols $\mathfrak{A}_1 \cong \mathfrak{A}_2$, if to every set E in either one of them there corresponds a set F in the other so that $E \doteq F$.

There are two common notions of separability for measure spaces. $\Omega(\mathfrak{B}, m)$

Received February 12, 1941.

[1] For a definition of the notions *field, Borel field, strict separability*, etc., to be used throughout this paper, see [3], pp. 752–753. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Thus if E and F are measurable sets we write $E \doteq F$ if $m(EF^{-1} + E^{-1}F) = 0$, where we use the notation E^{-1} for the complementary set $\Omega - E$.