UNSTABLE MINIMAL SURFACES OF HIGHER TOPOLOGICAL STRUCTURE

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1. **Introduction.** We are concerned with extending the calculus of variations in the large to multiple integrals. The problem of the existence of minimal surfaces of unstable type contains many of the typical difficulties, especially those of a topological nature. Having studied this problem for the case of one boundary [11], [12], we turned to the case of m boundaries. We discovered new difficulties not found either in the general theory when $m = 1$ or in the extensive minimum theory when $m > 1$. The case $m = 2$, however, appeared to contain the essentially new difficulties, and in order to present the relevant new ideas in their simplest form we have kept to this case.

We shall illustrate our results by a theorem which might have been conjectured by Newton. Let g_0 and g_1 be two parallel circles with planes orthogonal to their line of centers. Two such circles sufficiently near together bound a minimal surface of revolution of minimum area. This surface is generated by a segment of a catenary, and is always accompanied by another minimal surface of revolution not of minimum type. This classical result admits a simple generalization.

First recall that a simple, closed, rectifiable curve g is said to satisfy the *chord arc condition* 1a if the ratio of the length of an arbitrary chord of g to the minimum of the corresponding arc lengths of g is bounded from zero. A surface S is said to be a *disc surface* (*ring surface*) if S is given as the continuous image of a disc (circular ring). The above theorems on minimal surfaces of revolution admit the following generalization.

Let g_0 and g_1 be simple, rectifiable, closed curves in n-space satisfying the chord arc condition, separated by an $(n - 1)$-plane and possessing convex projections on suitably chosen $(n - 1)$-planes. If g_0 and g_1 bound a ring minimal surface belonging to a minimizing set, g_0 and g_1 also bound a ring minimal surface not of minimum type.

Our methods are based on the general critical point theory [11], [8]. We seek a function $W(P)$ defined on a metric space Π and of such a character that its critical points define ring minimal surfaces bounded by g_0 and g_1, or in the "restricted" case (see §7) define disc minimal surfaces bounded respectively by g_0 and g_1. To apply the general theory the function W should be boundedly compact, regular at infinity, and weakly upper-reducible in the sense of [11].

Received January 28, 1941; presented to the American Mathematical Society December 1939. See [10] and [13] for abstracts. Max Shiffman has written an independent paper on problems similar to the ones treated here. The methods he uses differ from those used here.