# THE DISTRIBUTION OF THE NUMBER OF SUMMANDS IN THE PARTITIONS OF A POSITIVE INTEGER

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1. It is well known that $p(n)$, the number of unrestricted partitions of a positive integer $n$, is given by the asymptotic formula [2][1]

$$p(n) \sim \frac{1}{4n3^{\frac{1}{2}}} \exp Cn^{\frac{1}{2}}, \qquad C = \pi(\tfrac{2}{3})^{\frac{1}{2}}. \tag{1.1}$$

In §2 we prove that the "normal" number of summands in the partitions of $n$ is $C^{-1}n^{\frac{1}{2}} \log n$. More precisely, we prove the following

Theorem 1.1. *Denote by $p_k(n)$ the number of partitions of $n$ which have at most $k$ summands. Then, for*

$$k = C^{-1}n^{\frac{1}{2}} \log n + xn^{\frac{1}{2}}, \tag{1.2}$$

*we have*

$$\lim_{n\to\infty} \frac{p_k(n)}{p(n)} = \exp\left(-\frac{2}{C}e^{-\frac{1}{2}Cx}\right). \tag{1.3}$$

The right member of (1.3) is strictly monotone and continuous; it tends to 0 as $x \to -\infty$ and to 1 as $x \to +\infty$. Hence, it is a distribution function. Also from (1.3) we clearly obtain the weaker result that if $f(n)$ is any function tending with $n$ to infinity, then the number of summands in "almost all" partitions of $n$ lies between

$$\frac{n^{\frac{1}{2}} \log n}{C} \pm f(n) \cdot n^{\frac{1}{2}}. \tag{1.4}$$

It is easily seen that the number of partitions of $n$ having $k$ or less summands is equal to the number of partitions of $n$ in which no summand exceeds $k$. Thus the preceding results can be applied to this case also.

In §3 we consider $P(n)$, the number of partitions of $n$ into unequal parts. (By a theorem of Euler, $P(n)$ is also equal to the number of partitions of $n$ into odd summands with repetitions allowed.) We obtain results similar to the above for $p_k(n)$, but we shall not give all details of the proof.

In §4 we derive an asymptotic formula for $p_k(n)$,

$$p_k(n) \sim \frac{\binom{n-1}{k-1}}{k!}, \tag{1.5}$$

valid uniformly in $k$ in the range $k = o(n^{\frac{1}{3}})$.

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[1] Numbers in brackets refer to the bibliography at the end of this paper.