INTERVAL-FUNCTIONS

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Introduction. Let T be a given set of open intervals I of the continuum, and let $\varphi(I)$ be a real function on T; we then say $\varphi(I)$ is an *interval-function* on the *range* T. We shall consider such interval-functions and point-functions readily associated with them. Certain of these point-functions are shown to be characterizable as upper or lower semi-continuous or as the monotone limit of a sequence of such functions. One of the associated point-functions is what we term the *kernel* of the interval-function. In general, this kernel is many-valued; of particular interest is the case where it is one-valued. We then call the interval-function *convergent*. It is shown that a necessary and sufficient condition for a point-function to be the kernel of a convergent interval-function is that it be the limit of a sequence of continuous functions. We prove furthermore, without reference to Baire's theorem, that the kernel of a convergent interval-function has a point of continuity on every perfect set with respect to the set;[1] and conversely, if a function $f(x)$ has a point of continuity on every perfect set with respect to the set, it is the kernel of a convergent interval-function. These two necessary and sufficient conditions yield a proof of the Baire theorem along lines different from those in the literature.

These results may be applied, for example, when the interval-function is chosen as follows. Let $f(x)$ be a given real function defined, say, on the entire continuum. We choose as the number $\varphi(I)$ associated with an interval $I = (a, b)$ the difference quotient $(f(b) - f(a))/(b - a)$. Allowing I to vary over the set of all intervals, we secure an interval-function $\varphi(I)$. When $f'(x)$ exists, $\varphi(I)$ is what we have termed convergent and $f'(x)$ is the kernel of $\varphi(I)$. Again, $\varphi(I)$ may be chosen as the upper boundary of a function $f(x)$ in the interval I, or as the saltus of $f(x)$ in I divided by the length of I. Other interval-functions may be obtained in a similar manner from a real function $f(x)$ by neglecting the values of $f(x)$ on sets which are negligible from the point of view of cardinal number, Lebesgue measure, etc. A number of interval-functions arise, also, in the study of a point-set S. For example, we may associate with an interval I the exterior Jordan or Lebesgue measure of S in I, or the relative exterior

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[1] We prove that such a point of continuity exists in the interior of the perfect set, i.e., the point is neither the left nor the right end-point of the set. In another paper *On the equation $dy/dx = f(x, y)$*, Bulletin of the American Mathematical Society, vol. 47, pp. 254–256, the author requires this form of the theorem.