FUNCTIONS WITH POSITIVE DERIVATIVES

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D. V. Widder has called my attention to the fact that a function having all its derivatives of even order non-negative in an interval is necessarily analytic there. This is a special case of results which have been stated, without detailed proof, by S. Bernstein;[1] it is useful in the theory of Laplace integrals.[2] In the first part of this note, I give an elementary proof of the theorem, and a proof, by methods differing from Bernstein's, of Bernstein's general theorem. This is

Theorem 1. *Let $\{n_p\}$ $(p = 1, 2, \cdots)$ be an increasing infinite sequence of positive integers such that n_{p+1}/n_p is bounded. If $f(x)$ is of class*[3] *C^∞ in $a < x < b$, and has the property that for each p $(p = 1, 2, \cdots)$ $f^{(n_p)}(x)$ does not change sign in $a < x < b$, then $f(x)$ is analytic in $a < x < b$.*

The greater part of this note is devoted to showing that Theorem 1 is, in a certain direction, the best possible result. I construct, for any sequence $\{n_p\}$ such that $n_{p+1}/n_p \to \infty$, a function whose n_p-th derivatives are positive in an interval, but which is not analytic in the interval. (The case where lim sup $(n_{p+1}/n_p) = \infty$, lim inf $(n_{p+1}/n_p) < \infty$ is left open.) More precisely, I prove

Theorem 2. *Let $\{n_p\}$ $(p = 1, 2, \cdots)$ be an increasing infinite sequence of positive integers such that $\lim_{p\to\infty} n_{p+1}/n_p = \infty$. Then there is a function $f(x)$, of class C^∞ in $-1 < x < 1$, such that $f(x) > 0$ in $-1 < x < 1$, and*

$$f^{(n_p)}(x) > 0 \quad (-1 < x < 1;\, p = 1, 2, \cdots), \tag{1}$$

$$f(x) \text{ is not analytic in } -1 < x < 1. \tag{2}$$

The function $f(x)$ will be defined by means of its development in a series of Chebysheff polynomials.

In Theorem 1, it could equally well be supposed that $f(x)$, instead of having n_p-th derivatives which do not change sign, is continuous and has n_p-th dif-

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[1] S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, 1926, pp. 196–197.

[2] D. V. Widder, *Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral*, Bulletin of the American Mathematical Society, vol. 40(1934), pp. 321–326.

[3] A function is of class C^n $(n = 1, 2, \cdots)$ if it has a continuous n-th derivative; of class C^∞ if of class C^n for every n.