A CLASS OF DIFFERENTIAL OPERATORS OF INFINITE ORDER, I

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Introduction. The present paper is the first part of an investigation devoted to the theory of differential operators of infinite order[1] of the form

$$G(\delta_z) = \sum_{k=0}^{\infty} g_k \delta_z^k. \tag{1}$$

Here

$$G(w) = \sum_{k=0}^{\infty} g_k w^k$$

is supposed to be an entire function,[2] the order and type of which will be subjected to various restrictions;

$$\delta_z = z^2 - \frac{d^2}{dz^2} \tag{2}$$

is the differential operator of Hermite-Weber; and $\delta_z^k = \delta_z \cdot \delta_z^{k-1}$. Putting

$$h_n(z) = (-1)^n e^{\frac{1}{2}z^2} \frac{d^n}{dz^n}(e^{-z^2}) \equiv e^{-\frac{1}{2}z^2} H_n(z), \tag{3}$$

where $H_n(z)$ is the n-th polynomial of Hermite, we find that

$$\delta_z h_n(z) = (2n+1) h_n(z). \tag{4}$$

The author has shown the importance of the differential operator $G(\delta_z)$ in the theory of Hermite series (see E. Hille [4]). There only those features of the theory were discussed which were of immediate use for Hermite series. In the present paper and its continuation we shall consider various questions omitted in the earlier discussion.

The basic notion of applicability of a differential operator was given on page 897 of the paper quoted above. Let $G(w)$ be a given entire function and let $\mathfrak{F}$ be a given class of analytic functions $\{f(z)\}$. We say that *the differential operator $G(\delta_z)$ applies to or is applicable to the class $\mathfrak{F}$ if the series*

$$G(\delta_z) \cdot f(z) = \sum_{k=0}^{\infty} g_k \delta_z^k f(z) \tag{5}$$

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[1] For a survey of the general field of differential operators of infinite order see R. D. Carmichael [1] and H. T. Davis [2]. The latter has an extensive bibliography. Numbers in brackets refer to the bibliography at the end of this paper.

[2] For the theory of entire functions used in this paper consult the treatise of G. Valiron [9].