THE METHOD OF ORTHOGONAL PROJECTION IN POTENTIAL THEORY

By Hermann Weyl

1. **Stating the problem.** Roughly speaking the classical boundary-value problem of potential theory for a region G in the Cartesian (x_1, x_2, x_3)-space consists in splitting a given function φ in G into two summands $\psi + \eta$, the first of which vanishes along the boundary of G, while the second η is harmonic. The two components are orthogonal if a metric in the functional space is based upon the Dirichlet integral

$$D[\varphi] = \int (\operatorname{grad} \varphi)^2.$$

$\int$ indicates integration over G.

This fact suggests the idea of replacing the scalar φ by the vector field

$$f = \operatorname{grad} \varphi \qquad \left(\text{in components: } f_i = \frac{\partial \varphi}{\partial x_i}\right)$$

and of operating in the Hilbert space of all vector fields f, with its metric defined by

$$\|f\|^2 = \int f^2 = \int (f_1^2 + f_2^2 + f_3^2).$$

Then the question arises how to characterize a vector field f as a gradient field without assuming more than its Lebesgue integrability. The vanishing of the line integral

$$\int (f \cdot dx) = \int (f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3)$$

over any closed curve in G will not do, because we have nothing but *spatial* integration at our disposal. The customary condition

$$\operatorname{rot} f = 0 \tag{1}$$

uses differentiation. Let v be any vector field vanishing at the boundary of G. The formula

$$\operatorname{div}\, [f, v] = (v \cdot \operatorname{rot} f) - (f \cdot \operatorname{rot} v) \tag{2}$$

Received July 13, 1940.