THE BINARY POLYHEDRAL GROUPS, AND OTHER GENERALIZATIONS OF THE QUATERNION GROUP

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1. Introduction. Hamilton's formulas

$$i^2 = j^2 = k^2 = ijk = -1$$

suggest the following definition for the quaternion group:

$$R^2 = S^2 = T^2 = RST \neq 1.$$

The natural generalization is

$$R^l = S^m = T^n = RST. \tag{1.1}$$

Let $\langle l, m, n\rangle$ denote the (largest) group defined by (1.1). This is symmetrical among l, m, n: for cyclic permutation, obviously; and for transposition, by changing R, S, T into T^{-1}, S^{-1}, R^{-1}, respectively.

Any two of R, S, T suffice to generate $\langle l, m, n\rangle$. For, if

$$R^l = S^m = T^n = RST = Z, \tag{1.2}$$

we can substitute $ZT^{-1}S^{-1}$ for R, obtaining

$$S^m = T^n = Z, \qquad (ST)^l = Z^{l-1}. \tag{1.3}$$

In particular, $\langle 2, m, n\rangle$ is simply defined by

$$S^m = T^n = (ST)^2. \tag{1.4}$$

Another definition for $\langle 2, m, n\rangle$ comes from the observation that $R = ST$. Substituting $S^{-1}R$ for T in (1.1), we obtain $R^2 = S^m = (S^{-1}R)^n$, or, writing S^{-1} for S,

$$R^2 = S^{-m} = (RS)^n. \tag{1.5}$$

In particular, $\langle 2, 2, m\rangle$ is the same group as $\langle 2, 2, -m\rangle$.

The relations (1.4) and (1.5) are reminiscent of Miller's[1]

$$s_1^m = s_2^n, \qquad (s_1 s_2)^2 = 1$$

and

$$s_1^2 = s_2^n, \qquad (s_1 s_2)^l = 1,$$

but are by no means identical with them.

Received June 20, 1940; presented to the American Mathematical Society, April 26, 1940.

[1] G. A. Miller, *Generalization of the groups of genus zero*, Transactions of the American Mathematical Society, vol. 8(1907), pp. 1–13.