THE APPROXIMATION OF IRRATIONAL NUMBERS BY FRACTIONS WITH ODD OR EVEN TERMS

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Introduction. We shall denote by (μ) the inequality

$$\left| \xi - \frac{A}{B} \right| < \frac{1}{\mu B^2}.$$

Recently, using a geometrical method of proof, W. T. Scott proved the following theorem.[1]

Let ξ be any irrational number, and let any one of the three types of fractions, of the forms odd/odd, odd/even, or even/odd, be selected. Then there are infinitely many fractions A/B of the required type which satisfy (1).

We shall give another proof of this theorem, making use of continued fractions. Furthermore, we shall prove that, *if two of the three types are selected, there are infinitely many fractions of one of these two types which satisfy* (2). These results should be compared with the older theorem of Hurwitz that *the inequality* ($5^{\frac{1}{2}}$) *can be satisfied, if approximations of all three types are allowed.*

Each of these theorems is the best result of its kind; that is, it is not always possible to satisfy (μ) with infinitely many approximations of the required sort, if $\mu > 1$, $\mu > 2$, or $\mu > 5^{\frac{1}{2}}$, respectively. This has been proved by Hurwitz and by Scott for the cases which they considered. For the unrestricted approximations, it is known that only those values of ξ whose continued fraction expansions have partial quotients which beyond a certain point are all equal to 1 do not admit infinitely many approximations satisfying (μ) for some $\mu > 5^{\frac{1}{2}}$, and in fact for $\mu = 2^{\frac{3}{2}}$. For his problem, Scott showed that for any $\mu > 1$ a ξ can be found such that (μ) cannot be satisfied by infinitely many approximations of the required type. We shall show that a ξ independent of μ can be found, and that indeed these exceptional values of ξ have the cardinal number of the continuum. A similar result is obtained in connection with the second problem. In both cases, an exact description of the exceptional irrationals is obtained in terms of the continued fraction expansion.

Finally, we shall solve the same problems in the cases in which we admit approximations of the type even/even in addition to some of the other types.

Continued fractions. We shall state here some results about continued fractions which are well known or easily proved. Let

$$\xi = [q_0, q_1, q_2, \cdots],$$

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[1] Scott, *Approximation to real irrationals by certain classes of rational fractions*, Bull. Amer. Math. Soc., vol. 46(1940), pp. 124–129.