# INVARIANTS OF SYSTEMS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

By Jack Levine

1. **Introduction.** As a first step in the discussion of ruled surfaces, Wilczynski[1] obtained the seminvariants and invariants of a set of two differential equations in two dependent variables of the form (2.1). Recently, Barnett and Reingold[2] have treated a similar problem for $n$ equations in $n$ dependent variables, and obtained sets of functionally independent seminvariants of order 1 for $n = 2, 3$, and 4, and an independent set of order 0 for general[3] $n$.

In this paper we consider the problem of obtaining seminvariants, invariants, and covariants of a general order $r$ of the system (2.1) containing $n$ equations in $n$ dependent variables. The methods of tensor analysis are used as these are suggested by the nature of the problem.

In §3 it is shown that every seminvariant is a function of quantities $P^i_j$ (defined in §2) and their covariant derivatives with components $P^i_{j;\alpha}$. By using the components $P^i_{j;\alpha}$ as independent variables, the number of equations in the complete system defining the seminvariants remains fixed for all orders $r$ and a given $n$. In the previous treatments the number of such equations increased with $r$.

In §7 a functionally independent set of seminvariants of general order $r$ ($>0$) is obtained for $n = 2$ and $n = 3$, and in §8, it is shown that the tensor seminvariants with components $P^i_{j;\alpha}$ form a complete set of such invariants and can be thus used in the equivalence problem of two systems (2.1).

2. **Seminvariants.** We shall consider the system of differential equations[4]

$$\frac{d^2y^i}{dx^2} + L^i_j(x)\frac{dy^j}{dx} + M^i_j(x)y^j = 0, \tag{2.1}$$

where $L^i_j$ and $M^i_j$ are arbitrary functions of $x$ possessing as many derivatives as are necessary for the purpose of our discussion. It is shown in W that the most general transformation of the $y$ which changes (2.1) into a similar form is

$$y^i = a^i_j(x)\bar{y}^j, \tag{2.2}$$

where $| a^i_j | \neq 0$, but the $a^i_j$ are otherwise arbitrary.

Received May 3, 1940.

[1] E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Chapter IV. We call this reference W.

[2] I. A. Barnett and H. Reingold, *Invariants of a system of linear homogeneous differential equations of the second order*, this Journal, vol. 6(1940), pp. 141–147. We refer to this paper as BR.

[3] See §2 of this paper for the definition of the order of a seminvariant.

[4] All indices throughout this paper have the range $1, 2, 3, \cdots, n$, unless otherwise stated. Also, we assume $n \geqq 2$ throughout.