ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES, III

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In this paper it is proposed to prove the following theorem.

THEOREM. *If $f(x)$ is such that at every point y on the closed interval $(-\pi, \pi)$ there are a function $g_y(x)$ and a $\delta > 0$ such that $g_y(x) = f(x)$ for $|x - y| < \delta$ and the Fourier series of $g_y(x)$ is absolutely summable $|C, 1|$, then the Fourier series of $f(x)$ is absolutely summable $|C, 1|$ on $(-\pi, \pi)$.*

This is analogous to a theorem on absolute convergence proved by Wiener.[1]

We must first make some general remarks about absolute summability $|C, 1|$. A series $\sum x_n$ is said to be absolutely summable $|C, 1|$ if

$$\begin{aligned}\sum_{n=1}^{\infty} |\sigma_n^{(1)} - \sigma_{n-1}^{(1)}| &= \sum_{n=1}^{\infty} \left| \frac{1}{n+1} \sum_{\nu=0}^{n} (n-\nu)x_\nu - \frac{1}{n} \sum_{\nu=0}^{n-1} (n-\nu-1)x_\nu \right| \\ &= \sum_{n=1}^{\infty} \frac{1}{(n+1)n} \left| \sum_{\nu=0}^{n} \nu x_\nu \right| < \infty.\end{aligned} \tag{1}$$

In order to apply this definition to Fourier series in the exponential form we set

$$x_n = (c_n e^{inx} + c_{-n} e^{-inx}).$$

It has been proved[2] that if a series $\sum x_n$ is absolutely summable $|C, 1|$, then

$$\sum_{1}^{\infty} \frac{|x_n|}{n} < \infty.$$

From this it follows that if a Fourier series is absolutely summable $|C, 1|$ over any interval (a, b), then

$$\sum_{-\infty}^{\infty} \frac{|c_n|}{n} < \infty. \tag{2}$$

By the Heine-Borel theorem and the hypotheses of the theorem there will be a finite number of overlapping intervals (δ_i, δ_i') covering $(-\pi, \pi)$ and functions $g_i(x)$ such that the Fourier series of $g_i(x)$ is absolutely summable $|C, 1|$ and $g_i(x) = f(x)$ on (δ_i, δ_i'). These intervals may be chosen so that $\delta_i < \delta_{i-1}' < \delta_{i+1} < \delta_i'$.

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[1] Norbert Wiener, *The Fourier Integral and Certain of Its Applications*, Cambridge University Press, 1933, p. 99, Lemma 6_{15}.

[2] E. Kogbetliantz, Bulletin des Sciences Mathématiques, (2), vol, 49(1925), pp. 234–256.