A CLASS OF FUNCTIONS BOUNDED IN THE UNIT CIRCLE

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1. **Introduction.** The principal object of this note is to show how continued fractions may be used to obtain inequalities for certain functions, particularly, for *moment generating functions* of the form

$$f(x) = \int_0^1 \frac{d\phi(u)}{1 + xu}, \tag{1.1}$$

where $\phi(u)$ is bounded and monotone non-decreasing in the interval $0 \leqq u \leqq 1$. To do this, we shall make use of results contained in three recent papers ([1], [3], [4]).[1]

2. **Some remarks on continued fractions.** It is easy to see that if $g_n \neq 1$ $(n = 1, 2, 3, \cdots)$, then the continued fraction

$$1/1 - g_1/1 - (1 - g_1)g_2/1 - (1 - g_2)g_3/1 - \cdots \tag{2.1}$$

and the series

$$1 + \sum_{i=1}^{\infty} \frac{g_1 g_2 \cdots g_i}{(1 - g_1)(1 - g_2) \cdots (1 - g_i)} \tag{2.2}$$

are equivalent. In fact, if A_n/B_n is the n-th approximant of the continued fraction, then $A_n = (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1})S_n$, $B_n = (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1})$, where S_n is the sum of the first n terms of (2.2). Consequently, if the g_n's are real and $0 \leqq g_n < 1$ $(n = 1, 2, 3, \cdots)$, the reciprocal of (2.1) is necessarily convergent, and hence the continued fraction

$$g_1/1 + (1 - g_1)g_2x_1/1 + (1 - g_2)g_3x_2/1 + \cdots \tag{2.3}$$

converges if $x_1 = x_2 = x_3 = \cdots = -1$ and is equal, for these special values of the x_n's, to

$$1 - \left\{1 + \sum_{i=1}^{\infty} \frac{g_1 g_2 \cdots g_i}{(1 - g_1)(1 - g_2) \cdots (1 - g_i)}\right\}^{-1}. \tag{2.4}$$

Moreover ([3], pp. 159–160), if $0 < g_1 < 1$ the continued fraction (2.3) converges uniformly over the domain

$$D: \quad |x_n| \leqq 1 \qquad (n = 1, 2, 3, \cdots);$$

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[1] The numbers in brackets refer to the bibliography.