EXISTENCE THEOREMS FOR BOLZA PROBLEMS IN THE CALCULUS OF VARIATIONS

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This paper is the third of a sequence. The first[1] paper of the three was devoted to the development of the theory of generalized curves originated by L. C. Young, and to the establishment of existence theorems for variational problems in which the curves admitted are generalized curves. In the second paper[2] we studied the necessary conditions satisfied by generalized curves which yield strong relative minima for Bolza problems. Here we shall combine the results of the preceding two papers to obtain existence theorems in which the minimizing curve found is a curve in the ordinary sense, and not a generalized curve. The theorems thus obtained are of considerable generality.

In §§1, 3 we set forth what might be called the every-day assumptions concerning the functions and curves involved in the problem. §2 contains some remarks about the relationship between generalized and ordinary curves. In §6 the first existence theorem is stated; its proof occupies §§4, 5, 6. A generalization of this theorem is established in §7. In §8 we establish a third theorem actually somewhat less general than that of §7, but having the desirable feature that its hypotheses are stated in terms of the data of the problem, without reference to the auxiliary problem of the minimizing generalized curve. The remaining two-thirds of the paper is devoted to the proof of corollaries immediately applicable to large classes of problems, and to the study of particular examples.

1. **Assumptions concerning the functions.** The principal object of study will be a functional

$$J(C) = g(y(a), y(b)) + \mathcal{F}(C) = g(y(a), y(b)) + \int_a^b f(y(t), \dot{y}(t))\, dt,$$

defined on a class K of curves $y = y(t)$ $(a \leqq t \leqq b)$ satisfying certain equations $\varphi^\alpha(y, \dot{y}) = 0$.

We now set forth our requirements on the nature of the functions g, f, φ^α and the class K of curves.

Throughout this paper we shall assume that

(1.1) $f(y, r)$ *and* $\varphi^\alpha(y, r)$ $(\alpha = 1, \cdots, m < \nu - 1)$ *are defined and continuous for all*[3] y *in a closed set* E *of* y*-space and for all* r, *and are positively homogeneous*

Received November 28, 1939.

[1] E. J. McShane, *Generalized curves*, this Journal, vol. 6(1940), pp. 513–536; henceforth cited as GC.

[2] E. J. McShane, *Necessary conditions in generalized-curve problems of the calculus of variations*, this Journal, vol. 7(1940), pp. 1–27; henceforth cited as NC.

[3] As before, y is a ν-tuple $(y^1, \cdots, y^\nu)$, and likewise r.