SUFFICIENT CONDITIONS FOR VARIOUS DEGREES OF APPROXIMATION BY POLYNOMIALS

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1. **Introduction.** Let E be a closed limited point set in the plane of the complex variable $z = x + iy$, and let the function $f(z)$ be defined on E. Let $\{p_n(z)\}$ be a sequence of polynomials in z of respective degrees n,[1] whose convergence to $f(z)$ is to be considered. The study of the relation between regions of convergence and geometric degree of convergence to $f(z)$ on E of the sequence $p_n(z)$ on the one hand, and regions of analyticity of $f(z)$ on the other hand, is in a relatively satisfactory state in the literature.[2] But the more delicate study of the relation between the degree of convergence both on E and elsewhere of the sequence $p_n(z)$ on the one hand, and the continuity properties of $f(z)$ *on the boundary* of the region of convergence on the other hand, has only recently been undertaken. It is the object of the present paper to contribute to this latter study.

To be more explicit, let the complement (with respect to the extended plane) K of E be connected, and regular in the sense that K possesses a Green's function $G(x, y)$ with pole at infinity.[3] Then the function $w = \varphi(z) = e^{G(x,y)+iH(x,y)}$, where $H(x, y)$ is conjugate to $G(x, y)$ in K, maps K conformally (not necessarily uniformly) onto the exterior of the unit circle γ in the w-plane so that the points at infinity in the two planes correspond to each other. We denote by C the boundary of E and also denote generically by C_ρ the locus $G(x, y) = \log \rho > 0$ or $| \varphi(z) | = \rho > 1$ in K. This notation is convenient in separating the study of approximation by polynomials into several problems.

Let the function $f(z)$ be assumed analytic merely in the interior points of E; *Problem α is the study of the relation between degree of convergence of $p_n(z)$ to $f(z)$ on E on the one hand, and continuity properties of $f(z)$ on C on the other hand.* In the real domain this problem has been extensively studied;[4] the complex domain has received less attention.[5]

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[1] A function which can be expressed in the form $a_0z^n + a_1z^{n-1} + \cdots + a_n$ is called a polynomial in z of degree n; we do not assume $a_0 \neq 0$.

[2] See for instance Walsh [1]; here and elsewhere numbers in square brackets refer to the bibliography at the end of this paper.

The reader is referred to that same work (Walsh [1]) for terminology not explicitly defined in the present paper.

[3] See for instance Walsh [1], pp. 65 ff.

[4] See for instance Bernstein [1, 2]; de la Vallée Poussin [1]; Jackson [1].

[5] See Sewell [1, 2, 3]; Curtiss [1, 2]. Jackson [2, 3] deduces the order of Tchebycheff approximation from approximation as measured by an integral.