APPROXIMATION TO FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS

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1. Let $f(x)$ be any integrable function. It is well known that there is just one trigonometric polynomial of order n which coincides with $f(x)$ in the $2n + 1$ points

$$x_i = i\frac{2\pi}{2n+1} \qquad (i = 0, 1, \cdots, 2n).$$

The explicit expression for these polynomials is very simple.[1] If we write

$$\varphi_n(t) = i\frac{2\pi}{n} \qquad (i = 0, 1, 2, \cdots, n-1) \tag{1}$$

for $2\pi i/n \leqq t < 2\pi\,(i + 1)/n$, the desired polynomials are

$$\begin{aligned} U_n(f, x) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\frac{\sin\,(n+\frac{1}{2})(x-t)}{\sin\,\frac{1}{2}(x-t)}\,d\varphi_{2n+1}(t) \\ &= \frac{1}{2n+1}\sum_{i=0}^{2n} f(x_i)\,\frac{\sin\,(n+\frac{1}{2})(x-x_i)}{\sin\,\frac{1}{2}(x-x_i)}, \end{aligned} \tag{2}$$

and plainly

$$U_n(f, x_i) = f(x_i) \qquad \text{for all } i.$$

In view of the close resemblance between (2) and Dirichlet's sum for the first n terms of the Fourier series of $f(x)$, we may expect to find some analogy between the behavior, for large n, of the polynomials U_n and the sums s_n of the Fourier series of $f(x)$. This is indeed the case. In some ways, however, interpolating polynomials have an advantage over partial sums. If the function is very smooth, polynomials give a much better approximation to the function than partial sums of the same order. Thus, if $f(x)$ is analytic, the respective errors after the n-th term are, as is well known, $(\frac{1}{2})^{n+1}4M/(n + 1)!$ and $M/(n + 1)!$ respectively, where M is the least upper bound of $|f^{(n+1)}(x)|$.

The advantage of the interpolating polynomials holds only for smooth functions. As the function becomes less and less smooth, the advantage decreases becoming ultimately a disadvantage. Thus Marcinkiewicz[2] and Grünwald[3]

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[1] See, e.g., E. Feldheim, *Théorie de la Convergence des Procédés d'Interpolation*, Mémorial des Sciences Mathématiques, no. 95, Paris, 1939, p. 22.

[2] *Interpolating polynomials for absolutely continuous functions* (in Polish), Wiadomosci Mat., vol. 39(1935), pp. 85–115; and *Sur la divergence des polynomes d'interpolation*, Acta Litterarum ac Scientiarum, vol. 8(1937), pp. 131–135.

[3] *Über Divergenzerscheinungen der Lagrangeschen Polynome*, Acta Litterarum ac Scientiarum, vol. 7(1935), pp. 207–221.