NOTE ON THE INVERSION OF THE LAPLACE INTEGRAL

By Harry Pollard

W. Feller and M. J. Dubourdieu have recently obtained the following simple inversion of a Laplace integral.[1]

If

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t) \tag{1}$$

converges for $x > c$ with $\alpha(t)$ normalized[2] and non-decreasing in every finite interval, then

$$\alpha(t) = \lim_{x\to\infty} \sum_{n=0}^{[xt]} \frac{(-x)^n}{n!} f^{(n)}(x) \qquad (t > 0). \tag{2}$$

By an obvious change of variable we can write this in the form

$$\alpha(t) = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(-1)^n}{n!} \left(\frac{k+\theta_k}{t}\right)^n f^{(n)}\left(\frac{k+\theta_k}{t}\right) \qquad (t > 0), \tag{2'}$$

where $\{\theta_k\}$ is a sequence[3] satisfying $0 \leqq \theta_k < 1$.

Earlier, however, Widder had obtained an inversion of (1) on the weaker hypothesis that $\alpha(t)$ is normalized and of bounded variation on every finite interval. His conclusion was[4]

$$\alpha(t) = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{(-1)^n}{n!} \left(\frac{k}{t}\right)^n f^{(n)}\left(\frac{k}{t}\right) \qquad (t > 0). \tag{2''}$$

By a comparison of (2′) and (2″) we are led to conjecture that the conclusion of Feller and Dubourdieu is not the best possible and that actually their operator (2) has the same degree of generality as the other. By methods closely related to Widder's we are able to establish this and even more, namely, that with the

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[1] W. Feller, *Completely monotone functions and sequences*, this Journal, vol. 5(1939), pp. 661–674; pp. 662–663.

M. J. Dubourdieu, *Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes*, Compositio Mathematica, vol. 7(1939), pp. 96–111.

These two authors obtained the result independently, Feller stating the equation in (2) only for points of continuity of $\alpha(t)$.

[2] I.e., $\alpha(0) = 0$, $\alpha(t) = \frac{1}{2}[\alpha(t+) + \alpha(t-)]$ $(t > 0)$.

[3] The sequence depends, of course, upon the particular value of t under consideration. In fact $\theta_k = xt - k$.

[4] D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, Transactions of the American Mathematical Society, vol. 36(1934), pp. 107–200. We shall refer to this paper as W. The particular result to which we make reference here is Theorem 2, p. 116. The form of the conclusion found there is only apparently different from (2″), as an examination of the proof will disclose.