THE PARTIAL DIFFERENTIAL EQUATION $\frac{\partial z}{\partial x} + f(x, y) \frac{\partial z}{\partial y} = 0$

By Louis D. Rodabaugh

Theorem 1. *Let g be a bounded, open, simply connected plane region. Let (x, y) be the rectangular Cartesian coördinates of a general point P of the plane for a particular coördinate system. Let $f(x, y)$ be a function*[1] *such that*

(a) *$f(x, y)$ and $f_y(x, y)$ are defined and are continuous in g;*

(b) *$f(x, y)$ and $f_y(x, y)$ have definite finite continuous limits on the boundary of g. Then there exists a function $\psi(x, y)$ such that in g:*

(a) *$\psi(x, y)$ is defined and is of class C′ with respect to x and y;*

(b) *$\psi(x, y)$ satisfies*

$$\frac{\partial z}{\partial x} + f(x, y) \frac{\partial z}{\partial y} = 0; \tag{1}$$

(c) *$\psi(x, y)$ satisfies*

$$\psi_y(x, y) > 0. \tag{2}$$

Proof. A solution curve of

$$y' = f(x, y) \tag{3}$$

shall be called a "characteristic" of (1). A known theorem assures that through each point of g there passes exactly one characteristic of (1), that these characteristics approach arbitrarily close to the boundary of g in both directions of the x-axis, and that they depend continuously on the initial point.

Remarks made by Kamke[2] for the case where g is an open, simply connected region "lying entirely in" an open region G apply also to the present case, and these remarks constitute the proof of

Lemma 1. *There exists a set of open, simply connected regions $q_1, q_2, \cdots$ with the properties:*

(a) *each q_n $[\equiv q(t_n)]$ is the set of points belonging to characteristics of* (1) *which lie in g and pass through an open finite vertical segment t_n lying in g;*

(b) *each h_n $[\equiv q_1 + \cdots + q_n]$ is an open region;*

(c) $q_1 + q_2 + \cdots = g$;

(d) *the common points of t_n and h_{n-1} form exactly one open segment, and t_n projects out of h_{n-1} in exactly one direction.*

Received September 1, 1939.

[1] Only real functions of real variables are considered in this paper.

[2] [1], §2, pp. 605–609. The numbers in brackets refer to the bibliography