SUPERPOSITION ON MONOTONIC FUNCTIONS

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The object of this paper is the study of the superposition of a function on a general monotonic function $y = g(x)$ or on an inverse of such a function, $x = g^{-1}(y)$, and the associated problem of what properties the set A can have when A' has a stipulated property and $g(A') = A$ or $g^{-1}(A') = A$. The first part of the paper deals with the functions $h(x) = f[g(x)]$ and $k(x) = f[g^{-1}(x)]$ when restrictions are put on $f(y)$, and $g(x)$ is any monotonic function. The results obtained are contrasted with the results for superposition on continuous functions, and are used to determine relations between classes of sets. The second part of the paper deals with the function $h(x) = f[g(x)]$ when restrictions are put on $g(x)$. The results obtained include a generalization of a well-known theorem concerning such superposition and give a good example of the Baire-measure duality. The paper closes with an application of the results to the Stieltjes integral.

The functions studied in this paper have as their domain of definition and range of values the interval [0,1]. The terminology is that of Kuratowski [3].[1]

1. A monotonic function considered as a transformation consists of a homeomorphism between two G_δ's, H and H', the relation $g(K) = K'$, and the relation $g^{-1}(K') = K$, where K' is a denumerable set of points and K is the sum of a denumerable set of points and a denumerable set of closed intervals (see [11]). To every point of K' corresponds one point or one interval of K, but the interval can be open, semi-closed, or closed.

A property P of sets is said to be a *restricted intrinsic invariant property* if any set homeomorphic to a set having property P also has property P, and if whenever a set X has property P, $G_\delta X + F_\sigma$ also has property P.

Theorem 1. *A general monotonic transformation and its inverse carry sets having a restricted intrinsic invariant property* P *into sets having the same property* P.

Let X have a restricted intrinsic invariant property P. Then

$$g(X) = g(XH + XK) = g(XH) + g(XK) = X'H' + g(XK).$$

Since X has a restricted intrinsic invariant property P and H is a G_δ, XH has a restricted intrinsic invariant property P. Since the monotonic function $g(x)$ is a homeomorphism between H and H', it transforms XH into a set $X'H'$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.