THE LAW OF LARGE NUMBERS FOR CONTINUOUS STOCHASTIC PROCESSES

BY J. L. DOOB

The purpose of this paper is to discuss the law of large numbers as applied to stochastic processes depending on a continuous parameter. In order to treat the temporally homogeneous processes, which will be discussed first, a form of the ergodic theorem of Birkhoff is proved which seems the from best suited to probability applications. In studying differential processes, some theorems on infinite series whose terms are independent chance variables will be needed. These are new and have some independent interest.

1. **The ergodic theorem and temporally homogeneous processes.** Let X be an abstract space, let $\mathfrak{F}_x$ be a Borel field of X-sets (including the space itself) and let $M(W)$ be a non-negative measure function defined on this Borel field. If $M(X) = +\infty$, we suppose that X is the sum of denumerably many sets of finite measure. Measurability of a numerically-valued function $f(x)$, defined on X, and integration are then defined in the usual way.

Let $f_0(x), f_1(x), \cdots$ be a sequence of measurable functions. Let n be any positive integer; let $E_1, \cdots, E_n$ be any Borel sets of numbers; let $h, \alpha_1, \cdots, \alpha_n$ be integers with $h \geqq 0$ and $0 \leqq \alpha_1 < \cdots < \alpha_n$; let W_h be the X-set determined by the conditions

$$f_{\alpha_j+h}(x) \in E_j \qquad (j = 1, \cdots, n). \tag{1}$$

Then if $M(W_h)$ is independent of h, the sequence $\{f_n(x)\}$ will be said to have the property H.

Let $f_t(x)$ $(0 \leqq t < \infty)$ be a one-parameter family of measurable functions. This family will be said to have the property H if the following condition is satisfied. Let $n, E_1, \cdots, E_n, h, \{\alpha_j\}, W_h$ be as above, except that $h, \alpha_1, \cdots, \alpha_n$ need not be integers. Then $M(W_h)$ is to be independent of h. If in either of the above two definitions, f_j (or f_t) is defined for $j = 0, \pm 1, \pm 2, \cdots$ (or $-\infty < t < \infty$) the definition of the property H remains as above, except that h need no longer be $\geqq 0$. It is evident that the integrability of any $f_j(x)$ in the above implies that of any other, and the value of the integral will be independent of the subscript.

The ergodic theorem of Birkhoff[1] is a theorem about a sequence of functions $f(x), f(Tx), f(T^2x), \cdots$, where $f(x)$ is x-measurable, and T is a measure-preserving point transformation. Such a sequence of functions has the property H.

Received January 14, 1939.

[1] A detailed discussion of the ergodic theorem and related theorems has been given by E. Hopf in his *Ergodentheorie*, Ergebnisse der Mathematik, vol. 5(1937), no. 2.