THE FIRST VARIATION IN MINIMAL SURFACE THEORY

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Introduction. We are concerned with harmonic surfaces S bounded by $n + 1$ non-intersecting simple closed curves[1] γ_k in a Euclidean m-space (x). The coördinates x^j on such surfaces, which are necessarily orientable and of genus 0, shall be harmonic functions of parameters (u, v) ranging over a connected region[2] B in the (u, v)-plane bounded by $n + 1$ circles $C_0, \cdots, C_n$. Let (r_k, θ_k) be polar coördinates in the (u, v)-plane with pole at the center (u_k, v_k) of C_k. Let $g_k(\theta_k)$ be a vector[3] defined on C_k giving an admissible representation of γ_k $(k = 0, 1, \cdots, n)$. Let $h(u, v)$ be the vector defining S. We suppose that $h(u, v)$ is continuous on the closure $\bar{B}$ of B and that its boundary values, represented in terms of the respective angular coördinates θ_k, determine admissible representations of the curves (γ) of the form

$$[g_0(\theta_0), \cdots, g_n(\theta_n)] = (g). \tag{0.1}$$

Conversely, each admissible representation (0.1) of the set (γ) determines a harmonic surface S defined over B. We admit no other surfaces.

Let $h(u, v)$ represent the harmonic surface defined by an admissible representation (g) of the curves (γ) and let $D(g)$ be one-half the sum (finite or infinite) of the classical[4, 5] Dirichlet integrals of the components $h^i(u, v)$ of $h(u, v)$. We term $D(g)$ the *Dirichlet sum.* The radii σ_k of the circles C_k and the coördinates (u_k, v_k) of their centers will be called the *circle parameters* of B. Set[6]

$$(\sigma_1, u_1, v_1, \cdots, \sigma_n, u_n, v_n) = (\eta).$$

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[1] We start with a 1-1 representation of γ_k on a circle C. We shall admit any other representation of γ_k which is obtained from the given representation by a monotone transformation (not necessarily 1-1) of C into itself.

[2] Although we restrict ourselves to surfaces of the topological type of B, the methods introduced are of such general character as to admit extension to the most general type of representation. For such extensions it is sufficient that the boundaries γ_k of the type regions B be circles. It is even sufficient that there exist a conformal map of a neighborhood of γ_k on B into an annular region, γ_k going into a circle under this conformal map. The introduction of subregions of multiply-sheeted Riemann surfaces naturally makes no difficulty.

[3] Curves and surfaces in our Euclidean m-space will invariably be represented as vectors as will the various Fourier coefficients which will presently enter.

[4] Jesse Douglas I: *Solution of the problem of Plateau,* Transactions of the American Mathematical Society, vol. 33(1931), pp. 263–321. Douglas II: *The problem of Plateau for two contours,* Journal of Mathematics and Physics, vol. 10(1931), pp. 315–359. Douglas III: *Minimal surfaces of higher topological structure,* Transactions of the American Mathematical Society, vol. 39(1939), pp. 205–298. Various additional references are here given.

[5] T. Radó, *On the problem of Plateau,* Ergebnisse der Mathematik, vol. 2(1933).

[6] We do not include the parameters (σ_0, u_0, v_0) in the set (η) because without changing the value of $D(g)$, C_0 can be taken as the unit circle with center at the origin.