THE NON-EXISTENCE OF A CERTAIN TYPE OF CONTINUOUS TRANSFORMATION

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1. It has been shown recently[1] that there exist continuous transformations defined on an arc A such that, if B denotes the image space, each point in B has *exactly* k inverse points in A ($k = 3, 4, 5, \cdots$). Thus for $k = 1, 3, 4, 5, \cdots$ it is possible to define an exactly $(k, 1)$ continuous transformation[2] on an arc. For $k = 3, 4, 5, \cdots$ the image may be taken to be a circle. We show that for $k = 2$ the circle cannot be an exactly $(k, 1)$ image of an arc, and in fact, no exactly $(2, 1)$ continuous transformation can be defined on an arc. This is the same as saying that it is impossible for any Jordan continuum to be so generated as the path of a moving point, whose coördinates are continuous functions of the time $(0 \leqq t \leqq 1)$, that every point of the continuum is passed through exactly twice.

2. In this section it will be shown that no exactly $(2, 1)$ continuous transformation exists carrying an arc into either an arc or a circle. First, we establish

LEMMA A. *Let T be a continuous transformation of A into B such that each point $b \in B$ has at most two inverse points in A. If $T(xy) = ab$ preserves end-points, where xy is an arc in A and ab is an arc in B, then T is topological on xy.*

This lemma can be proved immediately if we notice that it is essentially equivalent to the following theorem concerning a real continuous function:

If $f(x)$ is continuous for $0 \leqq x \leqq 1$, $f(0) = 0$, $f(1) = 1$, $0 \leqq f(x) \leqq 1$, and finally for each y $(0 \leqq y \leqq 1)$ $f^{-1}(y)$ consists of one or two values, then f is monotonic on $0 \leqq x \leqq 1$.

From Lemma A it follows easily that our image space can be no arc. Evi-

Received January 4, 1939; in revised form, May 31, 1939.

[1] The following example, formulated by G. E. Schweigert, shows that an exactly (3, 1) continuous transformation can be defined on an arc, the image being a circle.

Let C be a circle and (y_n) a sequence of distinct points on C in cyclic order with $\lim y_n = y_1$. Let the arc of C between y_n and y_{n+1} be denoted by Y_n. Set $x_n = 1 - n^{-1}$. Let the arc of the unit interval between x_n and x_{n+1} be denoted by X_n. Define T as follows. Map X_1 topologically on Y_1 with $T(x_1) = y_1$. For $n \geqq 1$, map X_{2n+1} on $Y_n + Y_{n+1}$ topologically with $T(x_{2n+1}) = y_n$. For $n \geqq 1$, map X_{2n} on Y_n with $T(x_{2n}) = y_{n+1}$. $T(\lim x_n) = y_1$.

If we add to the unit interval the semi-closed interval $1 < x \leqq 2$ and map it in one-to-one continuous fashion on the whole circle so that $T(1) = T(2)$, an exactly (4, 1) continuous transformation is obtained. The method obviously admits extension.

[2] By a $(k, 1)$ transformation is understood a single-valued transformation such that every point in the image space has at most k inverse points. An exactly $(k, 1)$ transformation is a single-valued transformation such that every point in the image space has exactly k inverse points.