THE CURVES OF A CONJUGATE NET

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1. **Introduction.** The purpose of this paper is to make some contributions to the projective differential geometry of the curves of a conjugate net on an analytic surface in ordinary space. §2 contains a summary of portions of the theory of a surface referred to a conjugate net. Power series expansions in non-homogeneous projective coördinates for the parametric curves on the surface are then computed to terms of the sixth degree. Some geometrical applications of these series are next made in a discussion of quadric surfaces having contact of the second order at a point of the surface. In the last two sections conjugate nets with specialized families are considered.

2. **Analytic basis.** In this section we indicate an analytic basis for the study of a surface referred to a conjugate net.

If the projective homogeneous coördinates $x^{(1)}, \cdots, x^{(4)}$ of a point P_x in ordinary space are given as analytic functions of two independent variables u, v by equations of the form

$$x = x(u, v), \tag{1}$$

the locus of P_x as u, v vary is an analytic surface S. If the parametric curves on the surface form a conjugate net, the four coördinates x and the four coördinates y of a point on the axis of the point P_x satisfy a completely integrable system of partial differential equations of the form[1]

$$\begin{aligned} x_{uu} &= px + \alpha x_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny \qquad (LN \neq 0). \end{aligned} \tag{2}$$

Let the point P_y be the harmonic conjugate of the point P_x with respect to the two foci of the axis. It is easy to verify that

$$y_u = fx - nx_u + sx_v + Ay, \qquad y_v = gx + tx_u + nx_v + By, \tag{3}$$

where we have placed

$$\begin{aligned} fN &= c_v + ac + bq - c\delta - q_u, & gL &= c_u + bc + ap - c\alpha - p_v, \\ -nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v, \\ sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p, \\ A &= b - (\log N)_u, & B &= a - (\log L)_v. \end{aligned} \tag{4}$$

Received May 16, 1939.

[1] E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, Chicago, 1932, p. 138.