ISOPERIMETRIC PROBLEMS OF BOLZA IN NON-PARAMETRIC FORM

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1. Introduction. For isoperimetric problems of the calculus of variations in parametric form an elegant proof of sufficient conditions for a strong relative minimum is afforded by the use of the so-called Lindeberg theorem. This result was obtained by Lindeberg [8][1] in 1909, who at that time applied it to the plane isoperimetric problem in parametric form.[2] Subsequently, this theorem was generalized and extended by Levi [7] and Tonelli ([13]; [14], vol. 1, p. 321). Recently Perlin [9] has developed a generalized Lindeberg theorem and applied it to the study of sufficient conditions for the parametric problem of Lagrange involving isoperimetric side conditions.

For non-parametric isoperimetric problems of the calculus of variations, however, neither the result obtained by Lindeberg [8], nor any one of the extended forms of the Lindeberg theorem established by Levi [7] and Tonelli ([13]; [14], vol. 1, p. 422), is effective in the proof of sufficient conditions for a strong relative minimum. Recently, Hestenes [5] has given a sufficiency proof for the isoperimetric problem of Bolza in non-parametric form. His proof is a generalization of the usual field method and does not use any analogue of the Lindeberg theorem; in particular, it involves the breaking up of the given extremal into suitable subarcs.

It is the purpose of the present paper to derive an effective Lindeberg theorem for non-parametric problems of the calculus of variations with isoperimetric side conditions. For the sake of generality, we consider specifically a problem of Bolza with variable end-points. It is of interest to note that the analogue of the Lindeberg theorem presented in §4 involves only the Weierstrass $\mathcal{E}$-function associated with the problem under discussion. It is first proved in §3 that if E is a non-singular extremal for the given problem satisfying the Weierstrass condition II_N, then there exists an associated problem which is entirely equivalent to the initial problem, and for which new and stronger forms of the usual Clebsch and Weierstrass conditions hold. The forms of these conditions thus derived render a simplicity to the results of §4 comparable to that for a problem which involves no auxiliary differential equations. They also enable one to simplify the expansion proof of sufficient conditions for the non-parametric problem of Bolza given by the author ([10], [11]).

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[1] Numbers in brackets refer to the bibliography at the end of this paper.

[2] See also Bolza [2], pp. 515–518.